DOCUMENT RESUME

ED 119 951 SE 019 772

TITLE A Teacher's Notebook: Mathematics, K-9.
INSTITUTION National Association of Independent Schools, Boston, Mass.
PUB DATE Sep 75
NOTE 148p.
AVAILABLE FROM National Association of Independent Schools, 4 Liberty Square, Boston, Massachusetts 02109 ($3.00)

EDRS PRICE MF-$0.83 Plus Postage. HC Not Available from EDRS.
DESCRIPTORS Curriculum; Elementary School Mathematics; Elementary Secondary Education; Instruction; *Instructional Materials; Junior High Schools; *Mathematics Education; Mathematics Materials; *Resource Materials; Secondary School Mathematics; *Teaching Guides; *Worksheets

ABSTRACT

This guide is divided into seven sections according to specific topics rather than by grade levels and/or grade level expectations. The topics encompass a K-9 program and include: numeration; measurement; operations and computational skills; algebra; informal geometry; sets, logic, and proof; and mathematical patterns. Each section lists concepts and objectives, references to resources, and materials used. In most sections detailed examples and comments on concepts to be developed are given. The guide contains an annotated bibliography of books for teachers and/or children. (JBW)

A TEACHER'S NOTEBOOK: MATHEMATICS, K-9

NAIS

National Association of Independent Schools

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Four Liberty Square, Boston, Massachusetts 02109

Printed in U.S.A.

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INTRODUCTION

Learning is an individual experience. No one can do it vicariously for someone else. The person doing the learning must be actively involved in whatever is going on to derive the greatest benefit from the situation. This is particularly true of children (and here we speak particularly of those in grades K-9) and must be taken directly into account when designing a mathematics curriculum for any age level.

Sitting still and keeping silent, being told what to do, how to do it, and when to do it are not natural, normal things for children. They must be free to move about the classroom, to discover and explore, to use the physical equipment available, and to discuss their findings with their classmates as well as with their teachers.

Children can and should work by themselves at times, and together with a partner or in small groups for some activities. Rarely are any two children ready for the same concept at the same time; even if they are, differences in the depth or degree of sophistication into which one or the other can stretch his thinking immediately crop up.

This kind of learning situation demands that the curriculum and the teacher involved be flexible and know the student well enough so that at any given moment the teacher can say, "Susan is ready for two-digit multiplication," or "Joe still needs more work with multibase blocks before he can abstract the idea of a base structure to just paper and pencil."

It means having a wealth of materials, physical and printed, on hand and ready for use at a moment's notice, and it means being ready with a suggestion for another approach if previous ideas fail or bog down. It means being willing to sit down alone with a child and listen instead of doing the talking.

It means encouraging original and creative thinking when it comes to problem-solving and helping children discover many ways to approach the same problem, even when it is totally contrary to anything we have known or used before. It means allowing children to help one another in these discoveries, since they often relate better and more actively among themselves than to an adult. It means being willing to admit mistakes and to enjoy the excitement of discovery with the child. Sometimes it means making deliberate mistakes or creating chaos out of which can come the most profitable learning situations. And it sometimes means allowing a child to "hang himself" with an idea before stepping in with a suggestion for correction.

Children are such individuals, and learning can be such a naturally exciting, stimulating, and rewarding experience for them when we, as teachers, show them that we have confidence in them--not only in their ability to learn but in their desire to do so. One book that should be in the hands of every mathematics teacher is Leonard Sealey's The Creative Use of Mathematics in the Junior School. Another is Freedom to Learn, by Edith Biggs and James MacLean.

Children seem to go through three distinct phases in their mathematical development: the exploration stage, the awareness stage, and the refining and mastering stage. They need time to discover, to explore, to play with physical equipment on their own. If given sufficient time to experiment and to verify their experiments by repetition at this concrete stage, the "awareness" of pattern and structure becomes intuitively a part of them: in some more than others, of course, but a very necessary skill to develop, no matter to what degree. The "refining and mastery" stage marks the transformation from the concrete and intuitive phases to the abstract form of mathematics: precise language, both written and spoken, writing and solving equations with an understanding and active use of the properties of mathematics, understanding the properties of geometric figures, spatial perceptions, and symmetries.

Where We're Headed Mathematically

Our aims must be to provide materials and guidance for children, to give them experience in working through tasks that lead to the kind of mathematical development outlined above, and to bring them to proficiency and understanding with numbers so that they may approach the more demanding programs of the upper grades with interest, excitement, and confidence.

Three very simple but powerful aims for teaching mathematics, stated by Biggs and MacLean at the beginning of Freedom to Learn, are (1) to free students, however young or old, to think for themselves; (2) to provide opportunities for them to discover the order, pattern, and relations which are the very essence of mathematics, not only in the manmade world, but in the natural world as well; and (3) to train students in the necessary skills. Three excellent, concise, and very important aims for our children. We hope the following curriculum guide will help fulfill them.

Creating a curriculum that covers all children, in all schools, is almost impossible. Levels of proficiency desired and needed will vary, as will ideas or concepts to be covered and aims to be met. Meeting a definite level by a definite time can be more damaging to a child's development than can be imagined. Of course, the adults responsible for the next level would like to see certain skills and understanding well in hand, but by now we are all well aware that children do not meet timetables in their development. Thus the question we must ask and answer honestly is, "Do we do what is best for each child, or do we meet the standards of the next grade, regardless of the readiness attained?" Massive readiness is the key to success at any stage; certainly, having fewer concepts and skills <u>well</u> in hand is better than a sketchy feel for many of them. One of the prime functions of the independent school is to develop each individual to his fullest potential, not to mold him into a precut pattern.

The broad topics we have chosen to cover allow for a tremendous amount of crossover learning and involve using skills and concepts from discoveries and explorations laterally as well as linearly. The topics allow for "cross pollination" of ideas and let children follow their natural inclinations as one idea opens up others to them.

Built into each discipline or area, and implicit in each one, is the need and use of the skills of arithmetic: addition, multiplication, subtraction, and division--to be developed to the depth that each child is ready to handle at any given time (there is not, and should not be, any one prescribed time for all children to "know their tables" or "basic facts" or be able to "do long division"); use of logic and efficient use of equipment in problem-solving; a creative approach to any problem (using a graph to represent findings); concepts of fractions--the whole and its parts, ratios--to be expanded to operations with rational numbers up to the level and limit of each child's ability; "accurate" estimation and using good, common sense, which means building on children's basic intuitive reasonings backed up by repeated investigations to verify "feelings." The teacher's part in this picture is to create problems in the various areas that stimulate not only the child's interest but keep his skill level constantly growing in balance with his background and thinking pattern.

On page 4 of <u>Freedom to Learn</u>, Biggs and MacLean say: "The pupil of the electronic age requires an educational environment that allows him maximum participation in discovery; schools must relate and synthesize rather than linearly fragment knowledge: they must provide a multiplicity of stimuli that will spark the child's curiosity and engender a continuing desire to learn." The more children can be helped to see that the same concepts and ideas and need for certain skills keep cropping up time after time, in the different projects they work on, the more natural their desire to acquire the skills will be.

How We Teach

There are as many "how to's" as there are children in your classes. And with all the exciting materials, blocks, games, puzzles, balances, scales, and the rest, it can be quite puzzling to know where to begin. One thing is sure: narrowly guided and strictly directed lessons with this kind of equipment can and will make mathematics, or any learning, just as dull and ineffectual as the "old" way--blackboard, chalk, "do it this way"--whatever that is. And one definitely would not use these things with primary children as one would with those in upper elementary and junior high grades.

With primary children, the stress surely has to be "play," experimentation ("play with it a while and see what it does"), touching, manipulating, trying things out, seeing what works and what doesn't, following an imaginative lead either from another child or from an adult, lots and lots of discussion among the children and between child and adult.

Finally comes the "ideal moment," when the children are ready for a specific question or problem that leads to understanding concepts and patterns and structure. Nobody can tell you exactly when that moment comes; you sense it when you work daily with children, watching their learning patterns and listening to their thought processes as they talk among themselves.

Several things that work quite well are explored here. But we cannot stress strongly enough that each teacher must develop a "style" and way of working that is exclusively his (or hers), one that he is comfortable with and finds most effective for the children he is working with at the time. You can, of course, learn a lot from a master teacher, but you can't be effective with children by "copying" a style that isn't you or by using prescribed questions from a teacher's manual written by someone thousands of miles away.

If you are most comfortable working with the whole class all at once, you can still turn this into a profitable session for all the levels of sophistication one encounters in a lesson. Use the concept or initial question or investigation as a point of departure for each child, letting him use what he has absorbed from these preliminaries to go as far as he can with the idea in his own way. This gives you a chance to look at the results and to plan the next working session to the child's best advantage. Once in a while you will come across two, or possibly three, children with the same learning problem; those few can be helped together and be encouraged to help one another.

Working with small groups--homogeneously or heterogeneously, whichever you prefer--is extremely effective. One really good step is to work with the small group of super "sharps," and then let them absorb the others into what they are doing, with them doing the "teaching."

If the amount of equipment in your room is small or limited in any way by a "shares" plan, one of the best ways around this is to work with that very small group of children and let them, in turn, work with others. Or set up three or four groups, each with a different "thing to work with" and a group project. You can walk around, ask questions, and give suggestions and encouragement about what is going on.

Recording, the biggest question with younger children, is best left to their imagination. Finding some way to tell you about what they are doing is a part of the problem they have to solve. Working together--whether in pairs, trios, or "tablefuls"--in a joint effort is one of the best situations for learning.

Whether you are working with younger or older children, one of the best approaches is a "problem"--an actual physical problem that needs solving: "The business manager is going to carpet the classroom and needs to know how much carpet to buy. How can we help him?" And don't be in too big a hurry to haul out the yardsticks. It's a great "first" exposure to area, so let them think about it for a while and ponder the details: what they know, what they need to find out, how to go about it, efficient and inefficient methods. The final answer is really insignificant compared with the "doing." With older children the outcome becomes more and more important, with such questions as, "Does that solution make sense?" "What do we need to be accurate?" "How accurate must we be?" "What previous knowledge do we need to work on this one?"

Often it is wise to start off with a statement that is true, maybe two or three, and then ask the children for another true statement that follows the same pattern or general idea and work on from there. Many, many ideas can be approached from a pattern idea, and once the pattern is mastered a more precise look at the concept is possible and profitable. What works all the time? Some of the time? Never? Looking for the underlying thread of "truth" provides clues and a basis for pursuing the "why." What children are ready to talk about they are ready to learn. If they are reluctant to take a chance, to venture an idea, to risk asking "Why?"--most likely they haven't had enough experience "doing" and "trying."

Some children work exceedingly well alone, and for them the project card is the ideal learning situation. The best project or "work" cards are the ones written by [illegible] for your children for whatever specific idea or concept you wish to explore. Some good ones are available commercially, but be careful that there isn't too much writing on the card, or too many details being "covered." And always be sure, with project cards, that you take the time to oversee the work and discuss the outcome with the child or children involved. All too often children work through an entire set of cards, answering all the questions correctly, yet manage to miss the "big idea." By carefully watching the work as it progresses, you can be alert to skills that may need reinforcing or even introducing, or an algorithm that can be "nailed down" right then, with your help.

We can all probably agree that a variety of working situations and lessons are most ideal for children. They need the occasional class or group lesson, time to explore and work on their own, and experiences with small groups. They need to listen, to learn from others, to contribute to the group, to withdraw and work out something for themselves. There is no one best way for everyone. Be alert, be aware, be a listener, be a leader, and don't be afraid to teach, if teaching is indicated, just as you mustn't be afraid to withdraw and allow learning to take place without you.

Teaching Goals

Our particular goals for teaching mathematics are strewn throughout this introductory section. And, as indicated above in "How We Teach," we as the authors would not totally agree on a set order for our goals. We would all undoubtedly agree on some goals that are not far removed from goals of the past nor, mostly likely, those to come: (1) skill with computation, to be sure, for all children, but commensurate with their own level of understanding and mathematical sophistication; (2) the ability to think and reason independently and creatively in all kinds of problem-solving; (3) reinforcing the child's curiosity for learning and helping to channel it into ever growing and expanding areas; and (4) feeling comfortable with and excited about all areas of mathematics--about all areas of learning. Specific goals for K-4, or 5-9, or K-9 are impossible to write or set a priority upon; that has to be done within your school, for your children, to meet the demands and expectations of their lives.

How To Use This Book

We have divided this book into seven sections according to specific topics rather than by grade levels and/or grade-level expectations. In each section we list the concepts to be developed, with brief references to excellent resources (expanded in the Bibliography) and, in some cases, some very detailed references (chapter and page number) on particular points. Where we feel there is no adequate resource available, or we have good ideas that have worked for us in our classes, we include special units for you to look over and try. Also, in each section, we list a collection of physical materials appropriate for use with the concepts being studied, and, where we have felt an additional need, a short list of objectives.

Basically, the seven topics covered are the ones we consider essential to a good, exploratory mathematics program at the K-8 level. These topics should be explored in a spiral manner, being introduced on a very simple level with the fives and sixes, and with increasing difficulty as the children progress. The best advice we can give is for you to use your own good judgment about where and when and how to introduce a concept, with lots of help and advice from the resources herein described. Increasing your own power with mathematics will certainly increase your effectiveness with the children.

We would strongly urge you to start by looking over and familiarizing yourself with the total "picture" of this book so that you can easily know where to look when you want help with a particular idea. Quite obviously materials, ideas, objectives, resources, even concepts, are going to pop up continually in cross references and overlappings: quite the picture of any good learning--not isolated ideas to be studied in isolation, but in constant use and reuse, overlap and "relate-iveness," with an intertwining of pattern and structure and "big idea."

We hope you will use this book as an IDEA CENTER. All the things we have listed, and some of our thoughts about how to use them, come from our experience that they work with children and that they give the kind of experience needed in the exploration stages that lead to the awareness and, later, to the refining and mastery levels.

Once again we say that this is not the only way. Each of us who has had a part in writing this book has his or her own style, a favorite way of teaching certain concepts, even separate priorities for which concepts are more important than others.

We hope this introduction invites you to use the balance of this book to find a new way to work with measuring, a different approach to fractions, a very different way to think about sorting and classifying, a really easy and logical method for subtraction or division, a new use for attribute blocks, or Cuisenaire rods, or geoboards.

We can't, as individual teachers, possibly be expected to think up all the creative uses for "stuff" in the classroom, especially if we are teaching in a self-contained one or in more than one subject-area discipline.

This book is a collection of ideas that work. We hope you will add to it yourself for your own future reference. We also hope you will pass along whatever good ideas you have. This Teacher's Notebook should continue to grow and expand, so don't hesitate to help "fatten" it. You can always reach us through NAIS, 4 Liberty Square, Boston, Massachusetts 02109.

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I

NUMBER, COUNTING, PLACE VALUE, AND NUMERATION SYSTEMS

A. Concepts and objectives

1. Distinction between counting and number
2. Not teaching place value until concept of number is firm
3. Counting a variety of discrete objects as opposed to counting Cuisenaire rods in a staircase: lots of counting
4. Partitioning of sets: counting
5. Developing place value: Cuisenaire rods, Dienes blocks, chip trading, counting frames and abaci, Papy mini-computer
6. Expanded notation
7. Scientific notation
8. Order of operations
9. Sorting: by attributes (color, size, shape, thickness, float, sink, long, short, thin, fat, etc.)
10. Matching: on a one-to-one correspondence
11. Ordering: from smallest to largest, shortest to longest, greater than, less than, more, less, same, smaller, smallest, large, larger, largest, etc.
12. Grouping: equal groups (addition, multiplication, subtraction, division, fractions, factors); unequal groups (addition, subtraction); place value and structure in many bases
13. Sharing: unequal partition (subtraction); equal partition (division, factors and fractions); multiplicative inverse
14. Counting: by 1's, 2's, 3's, etc.
15. Symbols: 0 1 2 3 4 5 6 7 8 9 + − · x ÷ $\overline{)\quad}$ □,△ □/△ < > $\neq$ $\perp$ $\equiv$ $\parallel$
16. Number names: one, two, three, four, etc.; Roman numerals; 7 − 6, 4 + 9, 8 ÷ 2, etc.
17. Cardinal/ordinal: difference in meaning and use

B. Materials

Transparent plastic boxes for storage

Sorting boxes (Gloucestershire sorting box)

String, rope, hoops, and colored card for enclosing rings

Familiar objects: cups and saucers, knives and forks, paintbrushes, pencils, balls, small blocks, beads, milk cartons, shells, spools, bottle caps, buttons, etc.

Pegboard and pegs

100-square board

Number-line materials (adding-machine tape) and strips

Nesting toys and interlocking cubes

Abacus (Japanese, Chinese, 10-bead counting frame, spike, multibase)

Numerals: plastic, wooden, written on cards

Beads with threaders

Trays with written numerals on which children can place the appropriate number of objects

Games: dominoes, Kalah, Lotto, Bingo, skittles, dice and dice games, spinning tops, etc.

Playing cards

Football and baseball cards with serial numbers

Cuisenaire rods

Stern arithmetic blocks

Color Factor blocks

Small packaged wooden/plastic toys: miniature cars (Corgi, Dinky, etc.), soldiers, horses, people

Attribute blocks, logic blocks

C. Resources (see Bibliography for complete references)

Charbonneau, <u>Work with the Sorting Box</u>

Cochran, <u>An Approach to Place Value Using Beans and Beansticks</u>

Davidson, Patricia, Chip Trading Activities

Dienes, Building Up Mathematics, Power of Mathematics

Fitzgerald et al., Laboratory Manual for Elementary Mathematics

Gattegno, For the Teaching of Elementary Mathematics

Goutard, Mathematics and Children

Nuffield Mathematics Project booklets

Page, Ways to Find How Many

Papy, Graphs and the Child, Mathematics and the Child

Williams and Shuard, Primary Mathematics Today

Comments on Concepts To Be Developed

I.A.1. Distinction between counting and number. Children often come to school able to count, that is, they say a sequence of words that we often mistakenly identify as the ability to count. Teachers often misread this form of "counting" as an indication that a youngster has a strong notion of number. Don't be fooled. (Some imaginative approaches to counting are described in Schools Council, Mathematics in Primary Schools; see Bibliography.) It is especially important to begin with discrete physical objects (as opposed to dots or pictures) and, further, to discern the level of concreteness or abstraction at which an individual student stands.

I.A.2. Not teaching place value until concept of number is firm. No comment needed.

I.A.3. Counting discrete objects vs. Cuisenaire rods. Often teachers start their work on number with Cuisenaire rods. What a disaster! There is very little inherent "fiveness" in a yellow Cuisenaire rod. Sure, it equals five white ones, but how well can this be seen by a child? How much better to work with buttons, beads, bottle caps, and other things to get a strong notion of counting and number before moving to the more sophisticated rods, whose quality of "number value" is associated with length instead of easier countability.

I.A.4. Partitioning of sets. Don't worry about "partitioning," which just means "sharing." Incidentally, you might want to consider avoiding set terminology altogether; many teachers are completely frustrated by "union," "intersection," "disjoint," "ordered," and other bits of terminology associated with sets. Forget all these fancy words. As part of your counting procedures, start by doing some informal adding, subtracting, multiplying, and dividing (yes, all at once, and in the primary grades), while talking about "sharing," "taking away," "combining," and so forth, and asking, "How many do you have left?" "How many does each of us have?" At some point, unobtrusively record what you and the students have been saying. Soon you'll simply be writing mathematical relationships, and the students will gain these "secretarial" skills by observation and imitation.

I.A.5. Developing place value. Developing a notion of place value is often very badly done, especially if teachers use the materials traditionally suggested by teachers' manuals. Fundamental to the notion of place value is the ideal of "exchanging" or "trading." One standard procedure is to use a counting frame. At some point we say that 10 red beads "are equal to" one blue bead (or whatever colors your particular counting frame calls for). No red-blooded child really believes this. Would you make the trade? The blue bead is the same size as the red bead. Which would you rather have, 10 red ones or one blue one? The youngsters usually acquiesce, However, even so, we must wonder whether anything has happened that contributes to the child's notion of place value. The model merely looks like the way we write numbers (or, more properly, "numerals," if we are going to insist on "proper" vocabulary, about which teachers worry unnecessarily).

One of the best materials to use at first is a set of Dienes Multibase Arithmetic Blocks (base-10 blocks are described below, in Section III). They come in a variety of bases--usually 3, 4, 5, 6, and 10. (There is considerable merit in having children work with blocks in bases other than 10. Most little ones count efficiently up to three; when counting up to 10, they may actually set aside nine or 11 things, and the words said and the hand movements often get "out of synch.") Most important, if 10 "units" are traded for one "long," you get the same amount of wood in exchange. If 10 "longs" are traded for one "flat," once again you get back the same amount of wood. They weigh the same and can be made to look the same. A possible substitute for Dienes blocks in this kind of work is a set of Cuisenaire rods with the prisms and cubes that are now available from the Cuisenaire Company.

If a youngster really understands trading and exchanging, he is probably ready for "chip trading" activities. Values assigned to the chips are arbitrary. The booklets and materials designed by Dr. Patricia Davidson (see Bibliography) for chip trading are excellent and can be nicely adapted for use with the more structured Dienes multibase blocks.

The Papy mini-computer is not hardware, but rather a method that uses paper and other very simple materials. It is especially useful in working on place value and doing computations.

<u>I.A.6-8. Notation and order of operations</u>. These items are adequately covered in most math textbooks. We would plead, however, that they not be introduced until there is a natural need for them.

<u>I.C. Resources</u>. All of the Nuffield Mathematics Project booklets are available from John Wiley and Sons, New York City. Every school with an elementary department should have a complete set of these. They were written <u>by</u> teachers <u>for</u> teachers. They cite many tasks that can be done with simple materials.

II

MEASUREMENT

A. Concepts and objectives

1. Time (clocks, calendars, stopwatches)

2. Weight: all kinds of scales, nonstandard and standard weights

3. Length (with nonstandard units first--hand, cubit, etc.--then English, metric)

4. Area and perimeter (nonstandard units first, intuitive approaches and other exploration before refining algorithms)

5. Volume and capacity: working with liquids and sand first, solids later; exploring concept of filling up well so that capacity and volume are well-fixed before moving into more formal considerations

6. Density: specific gravity (does it float?)

7. Measurement of angles as one way of talking about shapes

8. Speed

9. Temperature

10. Time, rate, and distance: actual experiments with inclined planes, trains, and stopwatches, clocking all sorts of activities

11. Surface area: as a "stamping" exercise with Cuisenaire rods and other physical materials; as a functional relationship, carefully planning for the number of variables to be handled at once; <u>initial exercise as a counting exercise as opposed to a plugging in of formulas</u>

12. Map-reading and scale-drawing, with a variety of road maps, airline maps, child-made maps, architectural drawings, etc.

13. Rounding off: number line; nearest inch, centimeter, etc.

14. See Section III on operations with rationals for work with David Page's "Upper Brackets, Lower Brackets"

B. Resources (see Bibliography for complete references)

Bell, <u>Mathematics in the Making</u> (see especially the booklets entitled <u>Looking at Solids</u>, <u>Pattern</u>, <u>Area and Perimeter</u>, and <u>Scale Drawing and Surveying</u>)

Biggs and MacLean, Freedom to Learn

Sealey, The Creative Use of Mathematics in the Junior School

Swartz, Measure and Find Out, Books 1-3

Tannenbaum and Stillman, Mapping

Williams and Shuard, Primary Mathematics Today

A Note about Metrication

Many teachers are in a quandary as to what they should be doing about metrication. Textbook manufacturers are making a bundle with new books on metrics. There is a rash of workshops on going metric. We would suggest that you stop worrying and dive in. Convert your math room or your science area to a place where measuring is done in metric units. Do you have a bathroom scale in metric units? How do you record temperature, distance, weight, volume, and so on? Don't get caught up in translating from our old units to metric units. Move into metric land.

TIME AND SPEED

Materials

Real clocks with clear faces placed at children's level

Toy clocks with gear wheels

Egg timers

Ten-second timer

Homemade water or sand clock

Candle clock

Homemade pendulum

Photographic clock with stopping device and large second hand

Stopwatches

Metronome

Pulsometer

Balls, toy cars, cylindrical pieces of wood, marbles, ball bearings to roll down inclined planes

Calendars

Day and date from daily papers to arrange in order

Radio and TV timetables

Bus, train, and airline schedules

State highway code booklets

Electric train

Battery-powered cars and boats

Anemometer--homemade or purchased

Objectives

Telling time: hours, minutes, seconds, elapsed time for radio and TV shows, time spent at play, lunch, dinner, at school, at sports, etc.

Modular arithmetic

Rate and ratio

Miles per hour, knots and nautical miles, kilometers

Computational practice with problems and projects related to the child's environment

AREA

Materials

Mosaic shapes of all types for pattern-making, etc.

Assorted shapes that fit together completely as well as those that do not

Inch squares (thousands!) and two-inch squares

Linoleum tiles of varying shapes and sizes

Floor and ceiling tiles

Carpet squares

Templates with squares, triangles, etc.

Transparent squares, rectangles, triangles, etc.

Pegboards

Geoboards

Sticks of different lengths

Bricks

Stern blocks, Cuisenaire rods, pattern blocks

Square feet, square yards, cut from mat board and colored cardboard

Squared paper, gummed squares, circles, rectangles

Maps

Objectives

Concept of area as way of measuring, as opposed to length; need for use of square units (both standard and nonstandard)

Inventive ways to figure areas of nonstandard shapes

Use of appropriate units for task at hand

Scale-drawing (ratio and proportion)

"Quick" way (algorithm) for finding areas of standard figures

LENGTH AND DISTANCE

Materials

Strings, ribbons, braids, wool, ropes, wire, etc.

Laths, canes, dowels, pegboard strips

Foot rulers: some marked, some unmarked, some marked in feet, some marked in inches, some in half inches and quarters

Yardsticks (some marked and some unmarked, as above) and meter sticks

Tape measures

Steel measures (tapes)

Surveyor's tapes and chains

Plastic-covered rope 22 yards long for homemade surveyor's chain

Trundle wheels: yard, meter, odd sizes

Calipers

Drinking straws, paper streamers, gummed paper for cutting into different lengths

Maps

Mileage markers

Hypsometer

45-degree set square

Objectives

Concept of length as unit of measure

Linear measure by nonstandard and standard units

Use of ruler, yardstick, meter stick, tape measure, etc.

Map-reading and interpreting scales: reproducing scaled maps

Perimeter

Appropriate use of centimeter, inch, foot, yard, meter, rod, kilometer, hand, cubit, mile, etc.

Ratio and proportion

CAPACITY AND VOLUME

Materials

Containers of assorted sizes and shapes, and some unusual ones, if possible

Rectangular boxes and cartons of all sizes

Cube boxes

Inch cubes, two-inch cubes, etc.

Centimeter cubes, liter cubes

Cylindrical containers (oatmeal and salt boxes)

Tins, bottles, beakers, jugs, cups, egg cups, medicine bottles, buckets, watering cans, basins, funnels, measuring spoons

Plastic tubes with stoppers, cut to lengths desired

Standard measures: pint, half pint, quart, gallon, liter, etc.

Cardboard milk cartons of all sizes

Sand, marbles, rice, dry cereals, coffee, tea, sugar, salt, dried beans, peas, etc., for weighing and measuring

Plasticine

Rubber bands for marking water levels

Large transparent geometric models with open side for filling

Objectives

Concept of how much something holds

Density versus specific gravity (not in formal terms!)

Ratio and proportion

Liquid measure and dry measure

Conservation (different shapes holding the same amount)

"Quick" way (algorithm) for finding volume of standard shapes

BALANCE AND WEIGHT

Materials

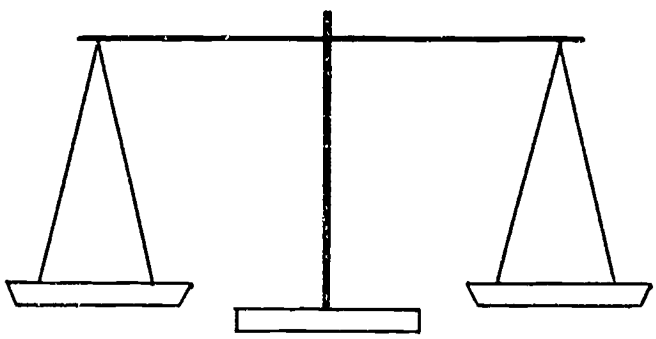
Flour, sugar, salt, tea, coffee, potatoes, dried beans and peas, dog biscuits, nails, screws, sand, clay, shells, stones, feathers, corks, marbles, plastic cubes, etc.

Clear plastic containers

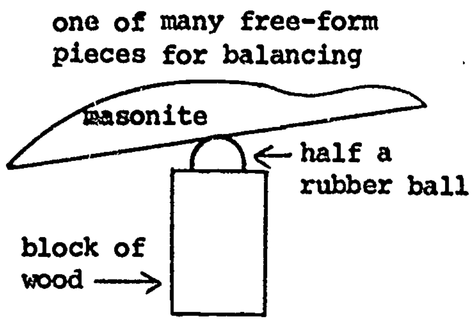
Scoops and funnels

Some parcels made up in exact weights: 6 oz., 4 oz., 100 grams, etc.

Scales of all types; simple balances are essential



homemade pan balance



Spring balances

Letter scales

Extension scales, such as the type used for fishing

Baby scales

Plentiful supply of weights: pounds, ounces, grams (brass weights, more accurate than other types, should be used for the smaller weights)

Objectives

Concept of weight as unit of measure

Concept of physical and numerical balance (in most cases, work with physics of balance should come before numerical balance)

More than, less than, equal to

Equations

Ratio and proportion

Gram weight

Pound weight

Density and specific gravity (not formally)

MONEY

Materials

Planks, cinder blocks, and cardboard cartons for setting up shelves and shops for playing store

Toys, candies, groceries, post office, cafe, milk bar, bank, etc.

Coins and money (to suit the group's ability and experience, as decided by the teacher)

Cash register or cash box

Gummed paper for pricing labels

Checks and checkbooks; savings-account passbooks and deposit and withdrawal slips

Chips and chip boards for chip-trading games

Objectives

Denominations and rate of exchange of money

Computational practice with money problems

Equal and unequal partitioning and grouping with buying and selling, making change, etc.

Use of checks and checking account

Decimal notation

III

EXTENSIONS OF THE NUMBER SYSTEM: OPERATIONS AND COMPUTATIONAL SKILLS

A. Concepts and objectives

1. Natural numbers

Operations with beans and other discrete objects, Dienes blocks, Cuisenaire rods, chip trading, Papy mini-computer, calculating devices (variety of abaci, mini-computers, hand calculators, adding machines, slide rules, desk calculators, Napier rods)

Discussion and exploration should focus on the nature of each operation, in particular the nature of subtraction and division, with differing options affecting the questions asked. Different approaches and skills should be encouraged and stressed.

2. Integers

Introduction through the number line, Pet Shop Stories, Pebbles-in-the-Bag, Postman Stories, nomographs, adding machines

Special attention should be paid to the role of the operator in multiplication. Units on Pebbles-in-the-Bag and Postman Stories are included in this section.

3. Rationals

a. Common fractions

Introduction through Cuisenaire rods: naming games ("If the dark green is one, what is . . . ?"); find a rod that can be broken into halves, into thirds, etc.; find a rod that can be broken into halves and thirds, etc.

Adding, subtracting, multiplying, dividing fractions

A special unit on operations with fractions is included here, with attention to the role of the operator in multiplication and special points in division, reading from right to left, the question to be asked, and doing division with the rods with color names.

b. Decimals

Introduced with Dienes base-10 blocks (reading, writing, and naming)

Addition: "carry" right away, using Dienes blocks, mini-computer, rods, chip trading, calculators

Subtraction: "borrow" right away, using same materials and procedures

Multiplication: rely on work with common fractions to develop the algorithm

Division

A special unit on decimals with Dienes base-10 blocks is included in this section.

c. Estimation: David Page's work, Do Something about Estimation; fractions and decimals in Page's "Upper Brackets, Lower Brackets"

d. Percent

4. Kye's arithmetic: a "different" way of looking at numbers and the operation of subtraction

Special short unit on Kye's arithmetic in this section

5. Irrationals

Introduction for discovery on geoboards; evolving rules for adding and multiplying; perimeters of polygons on geoboards; use of tables, slide rules, calculators

Special unit on irrationals included in this section with references for extra explorations

B. Materials

Cuisenaire rods, squares, and cubes

Dienes multibase blocks

Chip trading

Counting frames and abaci

Papy mini-computer

Unifix or Maxi-Linking cubes

Color Factor blocks

Beans and bean sticks

C. Resources (see Bibliography for complete references)

Brumfiel, Eicholz, and Shanks, Algebra I

Davidson, Jessica, Using the Cuisenaire Rods

Davidson, Patricia, Chip Trading Activities

Davis, Discovery in Mathematics, Explorations in Mathematics

Dienes, Building Up Mathematics, Power of Mathematics

Fitzgerald et al., Laboratory Manual for Elementary Mathematics

Nuffield Mathematics Project booklets

Page, Do Something about Estimation

Papy, Graphs and the Child, Mathematics and the Child

Williams and Shuard, Primary Mathematics Today

INTEGERS

Children move readily from natural numbers into signed numbers (integers) and, in fact, find these numbers far easier than "fractions," which usually come next in most texts. The following activities to introduce work with signed numbers are easily understood, fun, and readily available. We suggest this sequence.

1. Introducing integers. The following are possible alternatives:

   a. Number-line activities (Page, Number Lines, Functions and Fundamental Topics)

   b. Pebbles-in-the-Bag (Davis, Explorations in Mathematics, chap. 4)

   c. Pet Shop Stories (Davis, Discovery in Mathematics, chaps. 2 and 8)

2. Learning to add, subtract, and multiply integers

   a. Postman Stories (Davis, Discovery, chaps. 12 and 16, and Explorations, chaps. 5 and 6)

   b. Kye's arithmetic (Davis, Explorations, chap. 7)

3. Division of integers

   There are no Postman Stories to use as a model for division with integers. However, division comes quite naturally as an extension of multiplication. After using the Postman Stories for multiplication extensively, one can have the children discover the short-cut rule for multiplication and then see if that same rule works for division. (It does!)

4. Using integers (see item 5, below)

   a. Play tic-tac-toe (Davis, Explorations, chap. 2)

   b. Play Guess-My-Rule, using integers

   c. Work with open sentences, using signed numbers (equalities and inequalities)

   d. Graphing with signed numbers (Davis, Discovery, chap. 17)

   e. Quadratic equations: exciting work if children, working as a class, discover the "secrets" for quadratic equations (Davis, Explorations, chap. 10, and more in Discovery, chap. 13).

f. Nomographs (Meyer, Getting a Line on Math, and Pearcy and Lewis, Experiments in Math, Stage 2)

5. The work with integers we have outlined may begin at third or fourth grade. If the children have worked previously with tic-tac-toe, Guess-My-Rule, true-false-open sentences, and then move into the activities suggested here, they should have some understanding of the sorts of problems that follow. Use these for class work and then have the children invent problems of the same sort for their friends to try. This is often a very useful activity--for how can one "teach" or invent problems without extending one's own understanding?

Pebbles-in-the-Bag

(For background information, see Davis, Explorations)

1. After Ann says "go," Marvin puts in 5 pebbles and Joan takes out 3 pebbles. We would write (which would be correct?):

(a) 5 - 3 = -2

(b) 3 - 5 = +2

(c) 5 - 3 = +2

2. After Ann says "go," Gordon puts in 6 pebbles and Jerry takes out 8 pebbles. We would write (circle one):

(a) 8 - 6 = -2

(b) 6 + 8 = +14

(c) 6 - 8 = -2

3. What would be the action that preceded each of these notations in a game of Pebbles-in-the-Bag?

(a) 3 + 2 = +5	(d) 5 - 6 = -1	(g) 11 - 15 = -4
(b) 6 - 4 = +2	(e) 7 - 5 = +2	(h) 12 - 6 = +6
(c) 7 - 9 = -2	(f) 8 - 14 = -6	(i) 6 - 4 = +2

4. Write the notation for these actions and their results:

(a) After go, Ann puts in 8 pebbles and Jan takes out 6. ________

(b) After go, Jerry puts in 6 pebbles and Harry takes out 10. ________

(c) After go, Louise puts in 1 pebble and Sarah takes out 3. ________

(d) After go, Marvin puts in 6 pebbles and then puts in 4 more. ________

(e) Allan puts in 9 and takes out 7. ________

(f) Steve puts in 5 and takes out 9. ________

(g) Lori puts in 10 and takes out 4. ________

(h) Jennifer puts in 5 and takes out 12. ________

(i) Debbie puts in 25 and takes out 7. ________

(j) Dave puts in 14 and takes out 27. ________

5. Give the answer, using the Pebbles-in-the-Bag story if you need it.

(a) 11 - 15 =

(b) 8 - 3 =

(c) 4 - 7 =

(d) 3 - 10 =

(e) 4 - 16 =

(f) 7 - 23 =

(g) 6 - 19 =

(h) 12 - 25 =

(i) 8 - 21 =

(j) 4 - 6 - 8 =

Pebbles-in-the-Bag: Answer Sheet

1. (c) 5 - 3 = +2

2. (c) 6 - 8 = -2

3. (a) After Ann says go, Don puts in 3 pebbles and Betty puts in 2.

   (b) After Ann says go, Neal puts in 6 pebbles and Cindy takes out 4.

   (c) After Charlotte says go, Beth puts in 7 and Tony takes out 9.

   (d) After John says go, Bill puts in 5 and Margot takes out 6.

   (e) After Ken says go, Bruce puts in 7 and Marla takes out 5.

   (f) After Josh says go, Greg puts in 8 and Tim takes out 14.

   (g) After Ruth says go, Sandy puts in 11 and Tony takes out 15.

   (h) After Martha says go, Jean puts in 12 and Kurt takes out 6.

   (i) After Blair says go, Bill puts in 6 and Brookie takes out 4.

4. (a) 8 - 6 = +2

   (b) 6 - 10 = -4

   (c) 1 - 3 = -2

   (d) 6 + 4 = +10

   (e) 9 - 7 = +2

   (f) 5 - 9 = -4

   (g) 10 - 4 = +6

   (h) 5 - 12 = -7

   (i) 25 - 7 = +18

   (j) 14 - 27 = -13

5. (a) 11 - 15 = -4

   (b) 8 - 3 = +5

   (c) 4 - 7 = -3

   (d) 3 - 10 = -7

   (e) 4 - 16 = -12

   (f) 7 - 23 = -16

   (g) 6 - 19 = -13

   (h) 12 - 25 = -13

   (i) 8 - 21 = -13

   *(j) 4 - 6 - 8 = -10

*After John said go, Al put in 4 pebbles, Doug took out 6, and Tom took out 8. There were 10 pebbles less in the bag than before John said go.

Postman Stories

(For background information, see Davis, Discovery and Exploration)

1. Write appropriate number sentences to accompany the following Postman Stories:

a. A postman brings a check for \$3 and a check for \$5.

b. A postman brings a check for \$2 and a bill for \$7.

c. A postman brings a check for \$7 and takes away a check for \$4.

d. A postman brings a bill for \$3 and takes away a bill for \$9.

e. A postman brings two checks for \$3 each.

f. A postman takes away three bills for \$7 each.

g. A postman takes away two checks for \$3 each.

2. Compute the following:

*(a) +3 + +2 =

(b) +4 + -3 =

*(c) -3 + -2 =

(d) +7 + -11 =

(e) +7 + -7 =

(f) +6 - +2 =

*(g) +4 - +7 =

*(h) -5 - -2 =

(i) -4 - -9 =

(j) +4 - -6 =

(k) +4 - -2 =

*(l) -6 - +7 =

(m) -6 - +1 =

(n) +2 x +3 =

*(o) -2 x +3 =

*(p) +4 x -5 =

(q) -4 x -5 =

3. Write an appropriate Postman Story for each of the computations starred above.

4. (a) +15 ÷ -3 =

(b) -14 ÷ +7 =

(c) -16 ÷ -2 =

(d) +16 ÷ +2 =

Postman Stories: Answer Sheet

1. (a) +3 + +5 = +8
   (b) +2 + -7 = -5
   (c) +7 - +4 = +3
   (e) +2 x +3 = +6
   (f) -3 x -7 = +21
   (g) -2 x +3 = -6

2. (a) +5
   (b) +1
   (c) -5
   (d) -4
   (e) 0
   (f) +4
   (g) -3
   (h) -3
   (i) +5
   (j) +10
   (k) +6
   (l) -13
   (m) -7
   (n) +6
   (o) -6
   (p) -20
   (q) +20

3. (a) A postman brings a check for $3 and a check for $2.
   (c) A postman brings a bill for $3 and a bill for $2.
   (g) A postman brings a check for $4 and takes away a check for $7.
   (h) A postman brings a bill for $5 and takes away a bill for $2.
   (l) A postman brings a bill for $6 and takes away a check for $7.
   (o) A postman takes away two checks for $3 each.
   (p) A postman brings four bills for $5 each.

4. (a) -5
   (b) -2
   (c) +8
   (d) +8

RATIONALS

Common Fractions

Before adding and subtracting fractions with Cuisenaire rods, it is essential to work through a variety of activities to reinforce each student's knowledge of what fractions really are, and to extend each student's awareness of the relationships between different rods. Following are some suggestions for preparatory exercises.

Naming rods

Coding system

W = white	D = dark green
R = red	K = black
G = light green	N = brown
P = purple	E = blue
Y = yellow	O = orange

If red is called one, name the other rods:

R = 1	W = 1/2	Y = 2 1/2	N = 4
	G = 1 1/2	D = 3	E = 4 1/2
	P = 2	K = 3 1/2	O = 5

Certainly a variety of other names is possible. G might be labeled 3/2 instead of 1 1/2. With other rods, even more possibilities emerge.

Sample problems

If the white rod (W) = 1,	R =	G =	P =	Y =	D =
	K =	N =	E =	O =	
If the red rod (R) = 1,	W =	G =	P =	Y =	D =
	K =	N =	E =	O =	

If the light-green rod (G) = 1,	W =	R =	P =	Y =	D =
	K =	N =	E =	O =	
If the purple rod (P) = 1,	W =	R =	G =	Y =	D =
	K =	N =	E =	O =	
If the yellow rod (Y) = 1,	W =	R =	G =	P =	D =
	K =	N =	E =	O =	
If the dark-green rod (D) = 1,	W =	R =	G =	P =	Y =
	K =	N =	E =	O =	
If the black rod (K) = 1,	W =	R =	G =	P =	Y =
	D =	N =	E =	O =	
If the brown rod (N) = 1,	W =	R =	G =	P =	Y =
	D =	K =	E =	O =	
If the blue rod (E) = 1,	W =	R =	G =	P =	Y =
	D =	K =	N =	O =	
If the orange rod (O) = 1,	W =	R =	G =	P =	Y =
	D =	K =	N =	E =	

On all of these activities the teacher must be as patient as a saint and allow the children to work things out with the rods and must refrain from telling or indicating that "It is obvious that . . ."

Further activities

(a) Find a rod that can be divided into two equal pieces. (No saws or axes allowed.) Find another and another, etc.

red, purple, dark green, brown, orange

(b) Find a rod that can be divided into three equal parts.

light green, dark green, blue

(c) Find a rod that can be divided into fifths.

yellow, orange

An extension

(a) Find a rod that can be divided into halves and thirds.

dark green (Other answers are possible and equally acceptable. There is no need to find the smallest rod. The students may "build" rods larger than the orange one.)

(b) Find a rod that can be divided into thirds and fourths.

(After some time) "May we use more than one rod?" Teacher: "Sure."

orange plus red

On the following worksheet, please note the difference between the first six problems and those that follow. A rule emerges from problems 1-6 that can be extended to those that follow. Those who follow the rule will probably say "brown" for problem 7. Others will probably say "purple." Good discussions are possible. Once again, "May we use more than one rod?" Sure.

Find one rod that can be divided into both

1. halves and thirds. __________

   A white rod would be called __________.

2. thirds and fourths. __________

   A white rod would be called __________.

3. halves and fifths. __________

   A white rod would be called __________.

4. thirds and fifths. __________

   A white rod would be called __________.

5. fourths and fifths. __________

   A white rod would be called __________.

6. halves and sevenths. __________

   A white rod would be called __________.

7. halves and fourths. __________

   A white rod would be called __________.

8. halves and sixths. __________

   A white rod would be called __________.

Adding Fractions

On each of the following problems, first decide which rod should be one. Then name the white rod, and then the others.

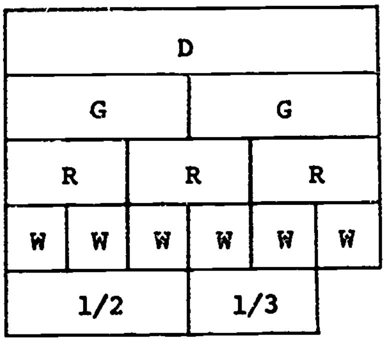
1/2 + 1/3 =

Dark green is one.

White rod is 1/6.

Light green is 1/2.

Red is 1/3.



D = 1

G = 1/2

R = 1/3

W = 1/6

1/2 + 1/3 = 5/6

In doing the worksheet, patterns begin to emerge after the first four or five problems. If students feel that they have a real clue about what is going on, they should be allowed to set the rods aside, to go off armed with their conjecture. One hopes they will be able to come up with the general form of #25 on Worksheet 1, which follows. Teachers should resist a passion for simplification and for lowest common denominator. For problem 11, 10/24 and 5/12 as answers from two different students should provoke a discussion of who is right and a resulting conclusion--"We're both right. They're the same thing."

The patterns that appear here are the same as those that appear in the work on quadratics section described in Davis, Discovery, chap. 13. Look for other extensions.

Adding Fractions: Worksheet 1

1. 1/2 + 1/3 = □
2. 1/3 + 1/4 = □
3. 1/2 + 1/5 = □
4. 1/3 + 1/5 = □
5. 1/4 + 1/5 = □
6. 1/2 + 1/7 = □
7. 1/2 + 1/9 = □
8. 1/2 + 1/2 = □
9. 1/2 + 1/4 = □
10. 1/2 + 1/6 = □
11. 1/4 + 1/6 = □
12. 1/2 + 1/8 = □
13. 1/4 + 1/8 = □
14. 1/5 + 1/10 = □
15. 1/6 + 1/8 = □
16. 1/6 + 1/10 = □
17. 1/5 + 1/10 = □
18. 1/2 + 1/10 = □
19. 1/3 + 1/6 = □
20. 1/3 + 1/9 = □
21. 1/6 + 1/9 = □
22. 1/5 + 1/2 = □
23. 1/4 + 1/3 = □
24. 1/5 + 1/3 = □
25. 1/□ + 1/△ =

On Worksheet 1 for addition of fractions, a general form was discovered:

$$1/\square + 1/\triangle = (\square + \triangle)/(\square \times \triangle)$$

Example: $1/2 + 1/3 = (2 + 3)/(2 \times 3) = 5/6$

This rule works perfectly well for problems 1, 8, and 9 on Worksheet 2, but will need refinement and improvement for the others. This technique of learning strategies might be called "benevolent torpedoing." The process of refinement can be very rewarding.

Students should be allowed to make mistakes. Mistakes are marvelous. They provide opportunities for beginnings of learning experiences.

A child who makes mistakes should be asked to show the teacher how to do a problem or two with the rods. One hopes new patterns will thereby emerge.

Often students will get 8/15 for #9 and 8/15 for #10 on Worksheet 2. The teacher might respond: "In #9 you're telling me that <u>one</u> third plus <u>one</u> fifth equals 8/15. How then can <u>two</u> thirds plus <u>two</u> fifths equal only 8/15?" Appropriate repairs come easily.

Note the new general form for #19 on Worksheet 2, which follows. On Worksheet 1 we found a general form for addition of fractions with numerators of 1. Now we are working on a general form for adding two fractions with the same numerators--not necessarily 1.

Some may worry about problems 1-7 on Worksheet 2. They may ask what these have to do with finding the general form for adding two fractions with the same numerators. We sneak in this added, simple generalization (FOR FRACTIONS WITH COMMON DENOMINATORS) because it's so easy to discover. Many kids say it's too easy and want to know when the "trap" is coming.

Adding Fractions: Worksheet 2

1. 1/3 + 1/3 = □

2. 2/7 + 3/7 = □

3. 4/11 + 2/11 = □

4. 2/4 + 1/4 = □

5. 3/△ + 4/△ =

6. △/7 + ▽/7 =

7. △ /  + □ /  =

8. 1/2 + 1/3 = □

9. 1/3 + 1/5 = □

10. 2/3 + 2/5 = □

11. 2/4 + 2/5 = □

12. 2/7 + 2/5 = □

13. 3/5 + 3/7 = □

14. 3/7 + 3/11 = □

15. 2/3 + 2/11 = □

16. 3/4 + 3/5 = □

17. 2/3 + 2/9 = □

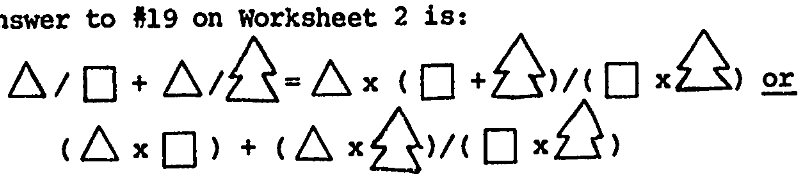
18. 4/5 + 4/7 = □

19. △ / □ + △ /  =

Now to head all the way home. Kids may well need to work out a number of these problems with rods again before patterns begin to emerge.

Once the series of addition worksheets is completed, the teacher should contemplate the process of mastery. Textbooks and worksheets should provide opportunities to perfect the required skills.

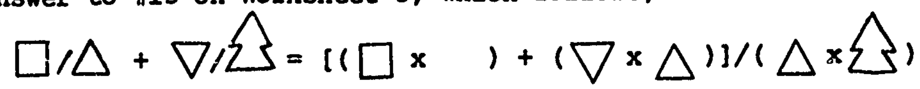
The answer to #19 on Worksheet 2 is:



$2/3 + 2/11 = 2 \times (3 + 11)/(3 \times 11)$ <u>or</u>

$[(2 \times 3) + (2 \times 11)]/(3 \times 11)$

The answer to #15 on Worksheet 3, which follows, is:



$2/3 + 5/7 = [(2 \times 7) + (5 \times 3)]/(3 \times 7)$

After some experience with the rods, and after some searching for patterns in adding fractions, students are ready to be challenged with becoming more efficient. They should not "do" addition of fractions all in one bite. Students should be allowed to become more sophisticated when they are ready for added sophistication.

Adding Fractions: Worksheet 3

1. 2/3 + 1/5 = □
2. 1/2 + 3/5 = □
3. 2/3 + 1/2 = □
4. 1/3 + 3/8 = □
5. 2/3 + 3/4 = □
6. 1/3 + 1/2 = □
7. 3/4 + 1/5 = □
8. 1/4 + 3/5 = □
9. 2/9 + 3/4 = □
10. 1/4 + 5/6 = □
11. 1/6 + 5/12 = □
12. 2/3 + 1/4 = □
13. 4/5 + 1/7 = □
14. 2/3 + 3/7 = □
15. □/△ + ▽/△ =

Subtracting Fractions: Worksheet

An extension of this work to subtraction is obvious. Try these in the same manner as you did above.

1. 1/2 - 1/3 = □
2. 1/3 - 1/4 = □
3. 1/3 - 1/5 = □
4. 2/3 - 2/5 = □
5. 2/5 - 2/7 = □
6. 2/3 - 2/11 = □
7. 3/4 - 2/3 = □
8. 3/5 - 2/3 = □
9. 2/3 - 1/5 = □
10. 3/4 - 1/3 = □
11. 1/□ - 1/△ =
12. 2/□ - 2/△ =
13. □/△ - ▽/🌲 =

Watch out for # 8!

Further work ought to include:

1. $3 - 1\ 2/3 = \square$

Let the light-green rod be 1.

White will be 1/3.

G	G	G	3 ones

You have 3. You are required to give away 1 2/3.

G	W	W	1 2/3

Giving away the one will be no problem. To give away two thirds, it will be necessary to trade 1 whole (1 light green) for 3 thirds (whites). Then the whole giving-away process is possible.

3	3 light greens
− 1 2/3	1 light green + 2 whites

Translation

2 3/3	2 light greens + 3 whites
− 1 2/3	1 light green + 2 whites

2. $3\ 2/5 - 1\ 1/2 = \square$

What rod is one? orange

What rod is 1/5? red   What rod is 1/2? yellow

You have 3 orange rods and 2 red rods.

You are required to give away 1 orange and 1 yellow.

In working with halves and fifths and using the orange rod as 1, the white rod is 1/10. Exchanges can now be made.

2 reds for 4 whites; 1 yellow for 5 whites; 1 orange for 10 whites

3 orange + 2 red = 3 orange + 4 white = 2 orange + 14 white

1 orange + 1 yellow = 1 orange + 5 white = 1 orange + 5 white

Translated into Numerals

3 2/5	=	3 4/10	=	2 14/10
− 1 1/2	=	− 1 5/10	=	− 1 5/10
				1 9/10

Students often come up with other interpretations. For example:

3 2/5 − 1 1/2

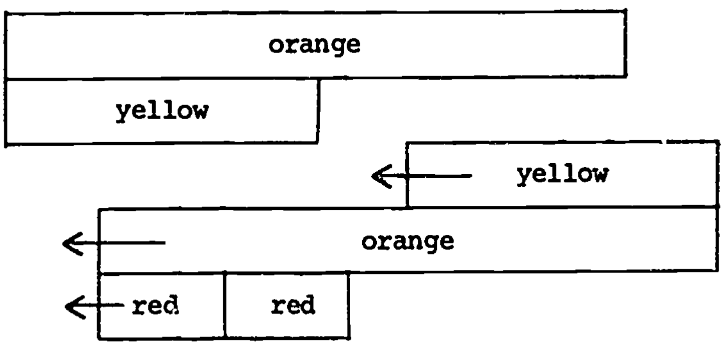
3 2/5 = 3 oranges plus 2 red

1 1/2 = 1 orange plus 1 yellow

Question: How much must be added to 1 1/2 to make 3 2/5? or

How much must be added to 1 orange + 1 yellow to make

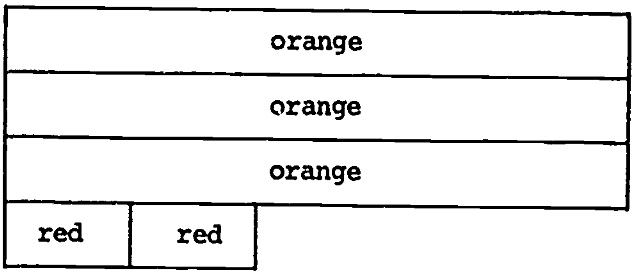
3 oranges + 2 red?



orange + yellow = 1 1/2

this added to orange + yellow will equal

(see next page)



You have added 1 orange + 1 yellow + 2 reds

Trading you get: 1 orange + 5 whites + 4 whites

1 orange + 9 whites or 1 9/10

Multiplying Fractions

Before working on multiplication with Cuisenaire rods, it is important to recognize the fundamental differences between the operations addition and subtraction and the operation multiplication. In adding the fractions 1/2 and 1/3, it is quite possible to represent both with rods. If the dark-green rod is chosen to be 1, then the light-green rod is 1/2 and the red rod is 1/3. These two can be placed in a train equivalent to 5 white rods. Since the white rod is 1/6, 1/2 + 1/3 = 5/6. Similarly, in considering 1/2 - 1/3, the difference in length between the light-green rod and the red rod is a white rod, or 1/6.

Multiplication is a very different matter. Consider 1/2 x 1/3. Only one of these fractions should be represented by a rod. The other rod is an operator. It tells you what to do to the rod. If we use the 1/2 as an operator, the 1/3 can then be a rod--the thing to be operated on.

1/2 x 1/3 means

Take 1/2 of 1/3, <u>or</u>

Take 1/2 of the red rod.

What is it? The white rod. The white rod is 1/6.

1/2 x 1/3 = 1/6

1/2 ⟶ operator

1/3 ⟶ a rod--the thing to be operated on.

Multiplying Fractions: Worksheet

1. Let the brown rod be 1. What rod is 1/2? What rod is 1/4? What rod is 1/8?

What is 1/2 of 1/2?	1/2 x 1/2 = ☐
What is 1/4 of 1/2?	1/4 x 1/2 = ☐
What is 1/2 of 1/4?	1/2 x 1/4 = ☐

2. Let the orange plus red be 1. What rod is 1/2? What rod is 1/3? What rod is 1/4? What rod is 1/6? What rod is 1/12?

What is 1/2 of 1/2?	1/2 x 1/2 = ☐
What is 1/2 of 1/3?	1/2 x 1/3 = ☐
What is 1/2 of 1/6?	1/2 x 1/6 = ☐
What is 1/3 of 1/4?	1/3 x 1/4 = ☐
What is 1/3 of 1/2?	1/3 x 1/2 = ☐
What is 1/4 of 1/3?	1/4 x 1/3 = ☐

3. Let orange plus purple be 1. What rod is 1/7? What rod is 1/2? What rod is 1/14?

1/2 x 1/7 = ☐	1/2 x 3/7 = ☐	3/7 x 1/2 = ☐
1/7 x 1/2 = ☐	1/2 x 5/7 = ☐	5/7 x 1/2 = ☐

4. Let orange plus yellow be 1. What rod is 1/3? What rod is 1/5? What do we call the white rod?

$1/3 \times 1/5 = \square$	$4/5 \times 1/3 = \square$
$1/5 \times 1/3 = \square$	$2/3 \times 4/5 = \square$
$2/3 \times 1/5 = \square$	$4/5 \times 2/3 = \square$
$2/5 \times 1/3 = \square$	

5. If you think you have solved the mystery of mysteries, communicate it by solving the following:

$1/\square \times 1/\nabla =$ $\qquad$ $\nabla/\square \times \spadesuit/\triangle$

$\nabla/\square \times 1/\triangle =$

* * * * * * *

Hopefully the students, after seeing the patterns emerge, will be able to arrive at the general form

$$\square/\triangle \times \nabla/\spadesuit = \frac{\square \times \triangle}{\triangle \times \spadesuit}$$

Extending this to 2 1/2 x 1 2/3 is essentially a renaming process and presents no great mystery. (One class called them "Kessler specials," named for a boy in the class, and they didn't mess around with "Kessler specials.") So 2 1/2 becomes 5/2, and 1 2/3 becomes 5/3. We can handle 5/2 x 5/3 with the general form we discovered above.

Dividing Fractions

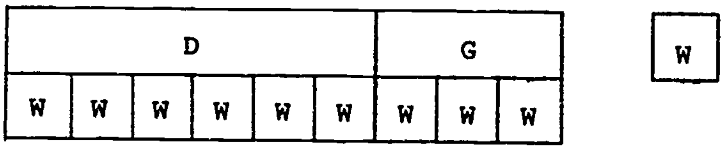
Several issues need to be considered before diving into division. First is a problem of reading:

$$1\ 1/2 \div 1/6 = \square$$

"1 1/2 divided by 1/6" is not very meaningful. Division problems should be read from right to left.

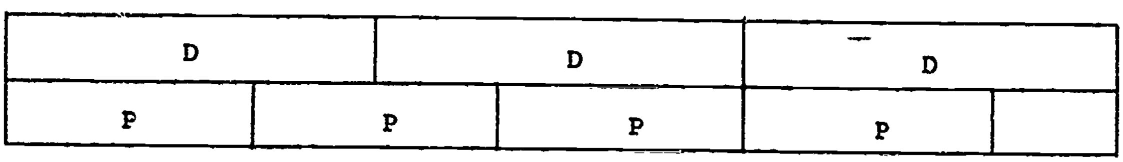
"How many times is 1/6 contained in 1 1/2?"

Or, translated into rods, "How many times is the white rod contained in a dark-green plus a light-green rod?"



Before working formally with division of fractions, it is wise to do a lot of division with Cuisenaire rods without using number names--just staying with the color names.

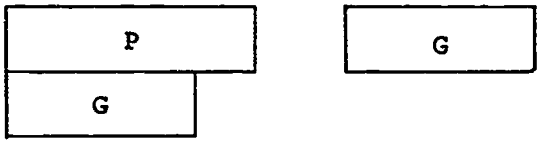
1. How many times is the red rod contained in the purple rod?
2. How many times is the purple rod contained in three dark-green rods? <u>4 times with something left over.</u>



What is this something left over? Half a purple rod.

How many times is the purple rod contained in three dark-green rods? 4 1/2 times.

3. How many times is a light-green rod contained in a purple rod?

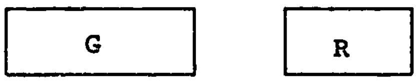


A whole light-green rod and 1/3 of a light-green rod.

How many times is a light-green rod contained in a purple rod? 1 1/3 times.

4. How many times is a white rod contained in a red rod?

5. How many times is a red rod contained in a light-green rod?



1 1/2 times.

6. How many times is a red rod contained in a brown rod?

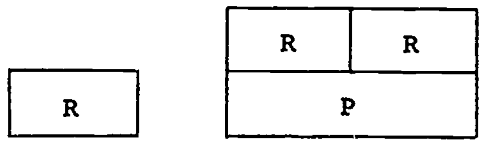
7. How many times is a purple rod contained in a brown rod?

8. How many times is a red rod contained in an orange rod?

9. How many times is a purple rod contained in an orange rod? Is this problem related to problem 8?

10. How many times is a dark-green rod contained in a blue rod? 1 1/2 times.

11. How many times is a purple rod contained in a red rod? You say your purple rod won't go into the red rod? What part of a purple rod will go into a red rod? 1/2 of a purple rod will go into a red rod.



12. How many times is a red rod contained in a white rod?

13. How many times is a dark-green rod contained in a red rod? You can't get a whole dark-green rod into a red rod. What part of a dark-green rod will fit? 1/3.

14. How many times will a blue rod fit into a dark-green rod? It won't fit? What part of a blue rod will fit in? 2/3 of it?

15. How many times will two oranges plus two reds fit into a purple rod?

16. How many times will a purple rod fit into an orange plus red?

Dividing Fractions: Worksheet

For this worksheet, consider orange plus red to be one. For each problem, choose the appropriate rods and read from right to left. How many times can a red rod be contained in orange plus red plus dark green?

1. 1 1/2 ÷ 1/6 = ☐
2. 2/3 ÷ 1/3 = ☐
3. 3 ÷ 2/3 = ☐
4. 2/3 ÷ 1/2 = ☐
5. 1/3 ÷ 1/6 = ☐
6. 2/3 ÷ 2 = ☐
7. 1/2 ÷ 1/3 = ☐
8. 1/3 ÷ 2/3 = ☐
9. 1 1/3 ÷ 1/3 = ☐
10. 1 1/3 ÷ 2/3 = ☐
11. 1 2/3 ÷ 2/3 = ☐
12. 1 2/3 ÷ 1/3 = ☐
13. 2/3 ÷ 3 = ☐
14. 3/4 ÷ 1/2 = ☐
15. 1/2 ÷ 3/4 = ☐
16. 2 2/3 ÷ 1 1/3 = ☐
17. 10/12 ÷ 5/6 = ☐
18. 1/3 ÷ 2 = ☐
19. 1/6 ÷ 1/3 = ☐
20. 1 ÷ 1/3 = ☐

Formalizing Division of Fractions

After kids have a firm notion of what dividing fractions is all about, it is time to get serious and head toward the conventional algorithms. A valuable concept to build is that of the multiplicative inverse. For example:

1. 1/2 x $\square$ = 1

   1/2 times what number is 1? Answer: 2

2. 1/4 x $\square$ = 1

   1/4 times what number is 1? Answer: 4

3. 1/3 x $\square$ = 1

   1/3 times what number is 1? Answer: 3

4. 4 x $\square$ = 1

   Four times what number is 1? Answer: 1/4

5. $\square$ x 3 = 1

   What number times 3 equals 1? Answer: 1/3

6. $\square$ x 3/2 = 1

   What number times 3/2 = 1?

   Let the kids work at this one. If at first you don't succeed, try, try again. If then you don't succeed, try a multiplication review sheet and include

   4/5 x 5/4 =

   3/2 x 2/3 =

Missing Factors: Worksheet

1. $\square \times 1/4 = 1$
2. $\square \times 1/2 = 1$
3. $\square \times 1/3 = 1$
4. $\square \times 1/100 = 1$
5. $\square \times 6 = 1$
6. $\square \times 4 = 1$
7. $\square \times 3 = 1$
8. $\square \times 17 = 1$
9. $\square \times 3/2 = 1$
10. $\square \times 5/3 = 1$
11. $\square \times 9/7 = 1$
12. $\square \times 2/3 = 1$
13. $\square \times 1/6 = 1$
14. $\square \times 3/8 = 1$
15. $\square \times 9/2 = 1$
16. $\square \times 4\ 1/2 = 1$
17. $\square \times 1\ 1/2 = 1$
18. $\square \times 1\ 2/3 = 1$
19. $\square \times 1\ 2/7 = 1$
20. $\square \times 5/4 = 1$
21. $\square \times 1\ 1/4 = 1$
22. $\square \times a/b = 1$
23. $\square \times v/z = 1$

<u>Curious fact, but helpful fact</u>:

6	÷	3	=	2
(6 x 5)	÷	(3 x 5)	=	2
30	÷	15	=	2
(6 x 10)	÷	(3 x 10)	=	2
60	÷	30	=	2
(6 x 1/2)	÷	(3 x 1/2)	=	2
3	÷	1 1/2	=	2

etc.

It turns out, in a division problem, that if I multiply both the dividend and the divisor by the same number (except for zero), the quotient will remain the same.

□	÷	△	=	⇧	
(□ x 3)	÷	(△ x 3)	=	⇧	
(□ x ▽)	÷	(△ x ▽)	=	⇧	(▽ ≠ 0)

Being inherently lazy, I prefer to divide by the easiest possible number. What is it? One, of course. Thus I always try to alter division problems so that I'm dividing by 1.

3/4 ÷ 1/2 =

What do I have to multiply 1/2 by to make it 1? [2]

(3/4 x 2/1) ÷ (1/2 x 2/1) =

6/4 ÷ 1 = 6/4 = 1 2/4 = 1 1/2

$$\frac{1}{2} \div \frac{3}{4} = \square$$

$$(1/2 \times 4/3) \div (3/4 \times 4/3) = \square$$

$$4/6 \div 1 = 4/6 = 2/3$$

We've arrived at a workable algorithm. Some refinements, done by kids, will provide the conventional form.

For alternatives to working with Cuisenaire rods, teachers might want to consider Stretchers and Shrinkers, a textbook set that stresses student involvement. Also effective is Fraction Bars, developed by Dr. Patricia Davidson and Albert Bennett, a combination of printed materials and manipulative materials for games and activities. See Bibliography for details.

It is always wise to have several alternative approaches. Some students respond to one better than another. Unfortunately many teachers resort to repeating one approach over and over again rather than trying something new on the student who does not understand.

DECIMALS

The following development of the notation used for writing fractions and mixed numbers in decimal form by means of the Dienes base-10 blocks can be used independently of or in conjunction with Dienes smaller-base blocks. We feel certain that students will have a clearer understanding of the ideas involved if they have been exposed to the earlier work. Both the blocks and the numbers involved are smaller, and the necessity for some means of distinguishing between the integral and fractional parts of a numeral emerges in a natural way.

However, we recognize that many students will not have had the desired exposure, nor will the teacher wishing to use the base-10 blocks always have the time to provide it. We have therefore prepared this unit to complement, but not to depend on, the earlier material.

Two basic ideas are required before beginning this work. Students who have had prior exposure to the blocks will be familiar with them, but students who are encountering them for the first time will need some introduction.

The first idea involves trading groups of one kind of piece for equivalent amounts of another piece, the object being to keep the same amount of wood while reducing the number of individual pieces to a minimum. Any time a person has more than 10 of any one piece, he must "trade" for one or more larger pieces. Thus 17 small cubes should be traded for one "long" and seven cubes; 23 longs would become two "flats" and three longs; and 49 flats would be exchanged for four "super-cubes" and nine flats. Other larger sizes can be made by combining: "super-longs" can be made by combining 10 10 super cubes; a "super-flat" is 10 super-longs; a "super-duper-cube" is made of 10 super-flats. (How many cubes in a super-duper-cube?)

The second idea is slightly more difficult but of equal importance. Again, students who have worked with the blocks before will recognize it, as will those who have used Cuisenaire rods to develop some of the concepts of fractions.

The idea involves naming a particular piece "one" and identifying others in terms of this original block. If we name the small cube "one," the long becomes "10," the flat "100," and the super-cube "1000." The number 368 would be represented by three flats, six longs, and eight small cubes.

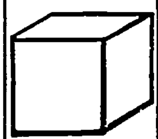
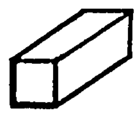
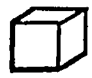
More examples arrive when larger pieces are named "one." For example, if a long is called "one," the small cube is 1/10, the flat 10, and the super-cube 100. Furthermore, 1/5 would be represented by two cubes, and 1/2 by five cubes. The larger the piece named "one," the smaller the fractions that can be represented. Sufficient practice in "naming" the pieces is essential before undertaking the operations.

Following are six worksheets with discussions of the ideas involved and a few sample problems. These are in no way expected to be exhaustive or even sufficient to provide mastery. They are simply a few minimal examples illustrating the basic concepts in the order in which we feel they should be presented.

Each teacher should make additional sheets, following the format shown or altering it to suit his own needs. Such sheets should supplement and extend the examples shown by providing new examples as well as practice on old examples.

Base 10: Worksheet 1

Since the small cube is named "one," the others become 10, 100, and 1000, respectively. On this worksheet, as on the five that follow, a "super-long" can be made by putting 10 super-cubes together, and its value is 10 times that of the super-cube, or, in this particular case, 10,000. Students should fill in the values of the other pieces at the top, as shown.

	10,000	1000	100	10	1
					
	super-long	super-cube	flat	long	cube
(a) three hundred sixty-eight			3	6	8
(b) one thousand twenty		1	0	2	0

Why some zeros and not others? Some youngsters might fill in the blanks opposite "three hundred sixty-eight" as 0 0 3 6 8. This is certainly as acceptable as 368. Some may write "one thousand twenty" as 0 1 0 2 0, while others write it as 1020. This difference in responses can produce a fruitful discussion: Which zeros are important? Which zeros can we omit and still say "one thousand twenty"? Which of the following are "one thousand twenty"?

01020

0102

0120

1020

120

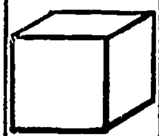
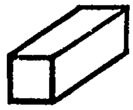
012

102

12

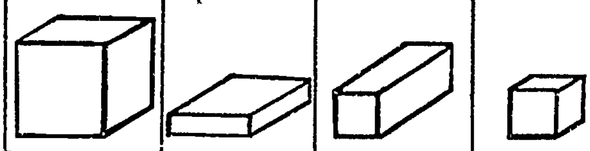
If some of the above don't say "one thousand twenty," what do they say?

Using the "10's" blocks, write figures for each of the numerals spelled out at the left.

					1
					
	super-long	super-cube	flat	long	cube
(a) sixteen					
(b) fifty-seven					
(c) one hundred forty-nine					
(d) three hundred seven					
(e) two thousand eighty-one					
(f) forty-three thousand seven hundred ninety					

Base 10: Worksheet 2

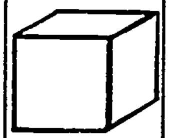
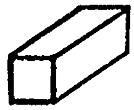
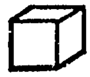
The long is now "one," so the cube is 1/10, the flat 10, and so on. In examples such as (c), below, be sure the student understands that in order to find three fifths he must first find one fifth.



	1000	100	10	1	1/10
	super-long	super-cube	flat	long	cube
(a) eighty-six and one half			8	6	5
(b) fifty-two and one fifth			5	2	2
(c) nine and three fifths				9	6

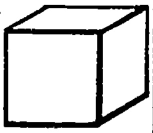
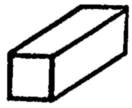
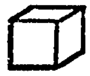
For those not readily able to see that eighty-six and one half is 86.5, how about some interim ones like "eighty-six and five tenths," "fifty-two and two tenths," "nine and six tenths"?

Using the "10's" blocks, write figures for each of the numerals spelled out at the left.

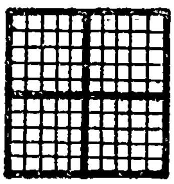
				1	
					
	super-long	super-cube	flat	long	cube
(a) nineteen					
(b) seven and one tenth					
(c) one hundred eight and one fifth					
(d) seventy-three and four fifths					
(e) six hundred twenty-one and six tenths					
(f) one thousand four hundred eight and a half					

Base 10: Worksheet 3

The flat is now "one," making the long 1/10 and the cube 1/100.

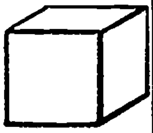
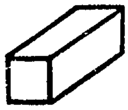
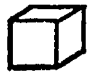
	100	10	1	1/10	1/100
					
	super-long	super-cube	flat	long	cube
(a) sixty-seven and a half		6	7	5	0
(b) twelve and one fourth		1	2	2	5
(c) sixteen and fifteen one hundredths		1	6	1	5

The decimal representation of 1/4 can be found in two different ways. Since the flat, which is "one," is made up of 100 small cubes, one fourth of it must be 25 cubes, which can be traded for two longs and five cubes. A more visual approach is to divide the flat into four equal squares, five units by five units,



any one of which can be rearranged into two longs and five cubes. Students should be allowed to use whichever approach seems the most natural. Some will prefer the numerical approach, while others will understand the geometric approach better.

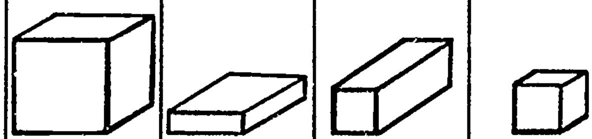
Using the "10's" blocks, write figures for each of the numerals spelled out at the left.

			1		
					
	super-long	super-cube	flat	long	cube
(a) seventy-three and twenty-seven hundredths					
(b) one and seven tenths					
(c) ninety-two and one fourth					
(d) three fourths					
(e) forty and four twenty-fifths					
(f) six hundred and fifty hundredths					

The concepts of the first three worksheets can now be extended by naming the super-cube "one," thereby enabling multiples of 1/8 to be identified in terms of the blocks. Some very nice geometric interpretations, similar to finding 1/4 above, emerge if we recognize that a super-cube can be made of eight base-5 super-cubes.

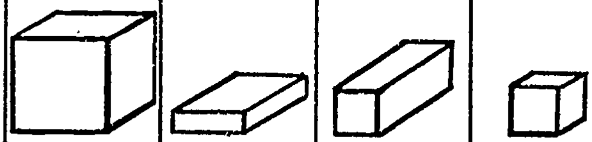
Base 10: Worksheet 4

The △ on this sheet indicates which block is "one." It is obvious that if these number_ had no designation for the "one" block, the numerals written would be meaningless in terms of relative values. The need for some notation (the decimal point) between the ones digit and the tenths digit begins to become apparent. If possible, allow the students to discover this themselves rather than pointing it out to them, since this idea will be extended even further in the following sheet.



	super-long	super-cube	flat	long	cube
(a) twelve and one half			1	2	5
(b) one hundred seventy			1	7	0
(c) two and seventeen hundredths			2	1	7

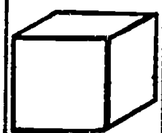
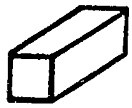
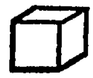
Using the "10's" blocks, write figures for each of the numerals spelled out at the left.



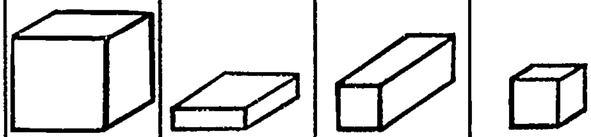
	super-long	super-cube	flat	long	cube
(a) six thousand four hundred twenty-one					△
(b) seven and one fourth			△		
(c) twenty and one half				△	
(d) seventeen and fifty hundredths			△		
(e) two hundred six and five tenths				△	
(f) nine and one eighth		△			

Base 10: Worksheet 5

The same sequence of digits on this sheet makes it even more obvious that we need to have a decimal point placed between the ones digit and the tenths digit to obtain the value of each numeral.

	super-long	 super-cube	 flat	 long	 cube
(a) four thousand seventy-five		4	0	7	5
(b) four hundred seven and one half		4	0	7	5
(c) forty and three fourths		4	0	7	5

Using the "10's" blocks, write figures for each of the numerals spelled out at the left.



	super-long	super-cube	flat	long	cube
(a) one hundred seventy-five					△
(b) seventeen and one half				△	
(c) one and three quarters			△		
(d) six hundred twenty-five					△
(e) sixty-two and one half				△	
(f) six and one fourth			△		
(g) five eighths		△			

Base 10: Worksheet 6

These examples are intended to provide additional practice in decimal notation.

A. Write the following mixed numerals as decimal numerals:

42 1/2 =

23 1/8 =

17 5/8 =

7/100 =

B. Write the following decimal numerals as mixed numerals:

7.25 =

19.3 =

127.08 =

.375 =

KYE'S ARITHMETIC

Robert Davis, in <u>Explorations in Mathematics</u>, describes a method of subtraction developed by a second-grade student named Kye who had worked with negative numbers:

$$\begin{array}{r} 42 \\ -\ 28 \\ \hline \end{array}$$

Instead of the usual "8 from 2 you cannot do," the youngster worked as follows:

$$\begin{array}{r} 42 \\ -\ 28 \\ \hline 2\bar{6} \end{array}$$

8 from 2 is -6 or $\bar{6}$

20 from 40 is 20

$20 + \bar{6} = 14$

These are neat kinds of numbers to work with. A few exercises in translation:

$6\bar{4} = 60 + -4 = 56$

$3\bar{2} = 30 + -2 = 28$

$4\bar{7} = 40 + -7 = 33$

$1\bar{2}1 = 100 + -20 + 1 = 81$

$\bar{7}2 = -70 + 2 = -68$

$\bar{6}\bar{8} = -60 + -8 = -68$

Now work these:

$2\bar{3} =$

$4\bar{6} =$

$12\bar{7} =$	$\bar{1}2\bar{3} =$
$1\bar{2}\bar{3} =$	$\bar{1}23 =$
$1\bar{2}3 =$	$\bar{1}\bar{2}\bar{3} =$

Operations with these numbers can be fun:

$$\begin{array}{r} 6\bar{2} \\ + \quad \bar{4}7 \\ \hline 25 \end{array} \qquad \underline{\text{Translation:}} \qquad \begin{array}{r} +58 \\ + \quad -33 \\ \hline 25 \end{array}$$

$$\begin{array}{r} 6\bar{2} \\ - \quad \bar{4}7 \\ \hline 10\bar{9} \\ 91 \end{array} \qquad \underline{\text{Translation:}} \qquad \begin{array}{r} +58 \\ - \quad -33 \\ \hline 91 \end{array}$$

$$\begin{array}{r} 6\bar{2} \\ \times \quad \bar{4}7 \\ \hline \bar{1}\bar{4} \\ 420 \\ 80 \\ \hline 2400 \\ \hline \bar{2}09\bar{4} \\ \bar{2}086 \\ -1914 \end{array} \qquad \begin{array}{r} -2000 \\ 86 \\ \hline -1914 \end{array} \qquad \underline{\text{Translation:}} \qquad \begin{array}{r} +58 \\ \times \quad -33 \\ \hline 174 \\ 174 \\ \hline -1914 \end{array}$$

The above is a good chance for drill with both natural numbers and signed numbers.

A good "Noodle Needler" is the unusual numeration system found on page 41 of Fitzgerald, Laboratory Manual for Elementary Mathematics.

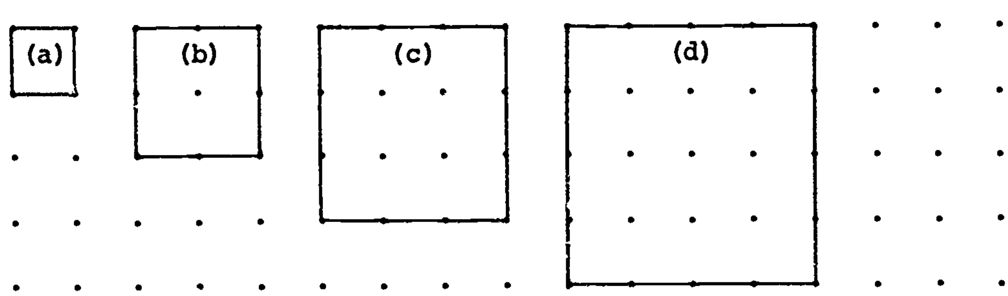
IRRATIONALS

Irrationals need to be introduced to children in a mathematically valid manner that will clearly show that $\sqrt{2}$, $\sqrt{3}$, etc., are new and different numbers that can be added and multiplied and so on. These numbers were originally discovered by the Pythagoreans and other early Greeks through geometry. We can recreate this bit of mathematical history with children. If you find this an intriguing idea, try this sequence of activities (which was suggested by some work done with second graders by Professor Alice Hart, of the University of Illinois at Chicago).

I. The first step needs to have the children working as a group, each with his own geoboard. Assuming that the children have done previous work finding areas on the geoboard, pose these questions for the class:

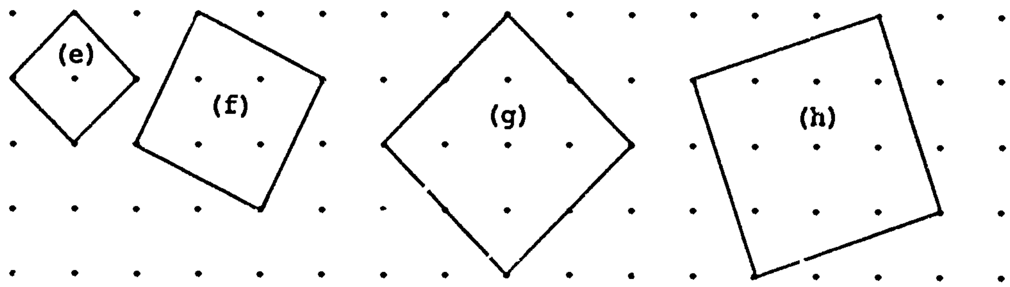
1. What is the smallest square you can make, using one rubber band?
2. What is the largest square you can make with one rubber band?
3. How many different (in area) squares can you make with one rubber band?

II. Then allow the children to work on these questions in small groups or alone as they choose. At an appropriate time discuss your findings.



Area is (a) 1 square unit (c) 9 square units

(b) 4 square units (d) 16 square units



Area is (e) 2 square units (g) 8 square units

(f) 5 square units (h) 10 square units

III. We've played Guess-My-Rule many times, so let's use it to help us find a relation between the area of a square and its side. First, let's collect the data about our squares and put it in a table.

measure/area of square $\longrightarrow \square$

m/side of square $\longrightarrow \triangle$

$\square$	$\triangle$	
1	1	(by counting)
4	2	" "
9	3	" "
16	4	" "

IV. At this point the teacher can ask the children whether the m/sides might be given another name.

$\square$	$\triangle$	
1	1	might be $\sqrt{1}$
4	2	" " $\sqrt{4}$
9	3	" " $\sqrt{9}$
16	4	" " $\sqrt{16}$

Is there a relation between the area of the square and the side of the square? Can you state it in words? [The side is the $\sqrt{}$ of the area.] Can you write a mathematical sentence that expresses the relation? [$\sqrt{\square} = \triangle$]

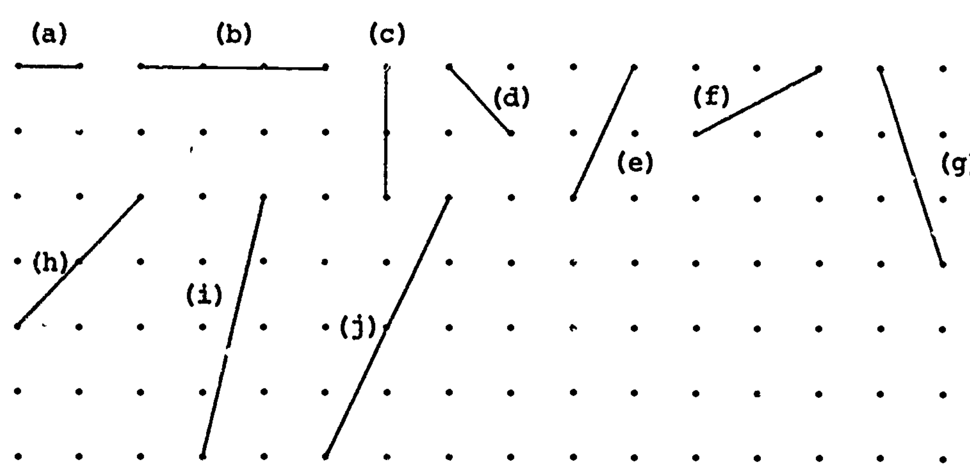
V. With the remaining four squares--(e) through (h) in step II, above--the question arises as to what name we should give the sides. Since we cannot determine it by counting, we must use the relation we've discovered. The teacher must judge how much discussion or supplementary work is needed here.

Completing our table, we now have:

□	△	
1	1 or $\sqrt{1}$	
4	2 or $\sqrt{4}$	
9	3 or $\sqrt{9}$	
16	4 or $\sqrt{16}$	
2	- $\sqrt{2}$	
5	- $\sqrt{5}$	no counting number names
8	- $\sqrt{8}$	
10	- $\sqrt{10}$	

VI. Now give the children problems like these:

Find the length of each of the following line segments. If in doubt, construct a square whose side is the given line segment and use the relation you have discovered. You may want to use more than one geoboard.



Find two different names for (h) and (j), above.

Noodle Needler:

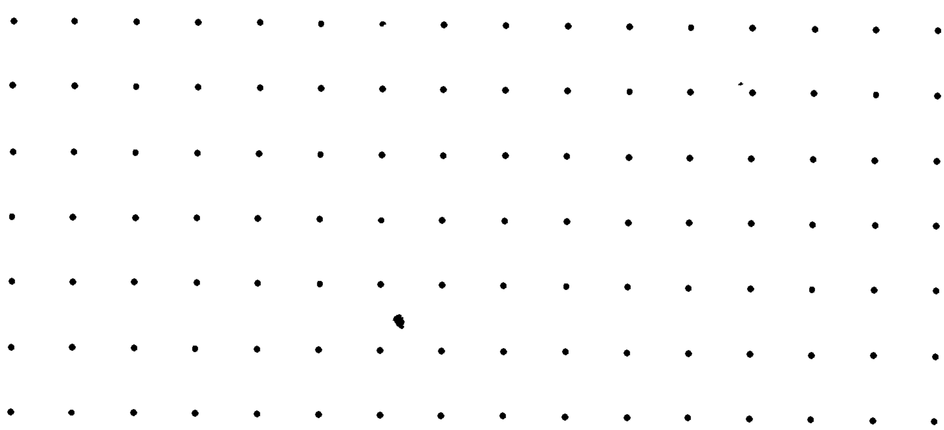
(1) How many different (in length) line segments can you make on your geoboard? Draw them on dot paper.

(2) Find the length of each.

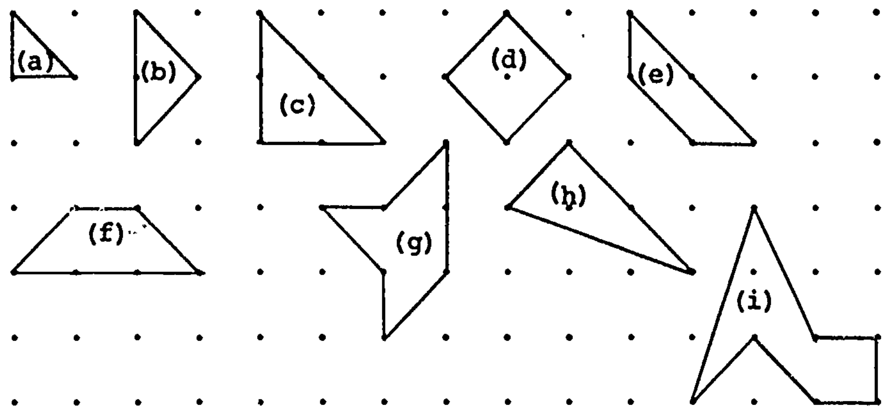
Answers are on page 83.

VII. Or problems like these . . . On your geoboard try to make line segments of the following lengths. (You may want to use more than one geoboard.) Record your results on the dot paper below. Answers are on page 84.

(a) $\sqrt{1}$	(d) $\sqrt{17}$	(g) $\sqrt{8}$	(j) $\sqrt{10}$
(b) $\sqrt{5}$	(e) $2\sqrt{2}$	(h) 2^2	(k) $2\sqrt{5}$
(c) 5	(f) $\sqrt{20}$	(i) $4\sqrt{2}$	Noodle Needler: $\sqrt{32}$



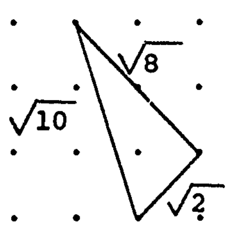
VIII. Or like these . . . Give the perimeter of these polygons. Use your geoboard if you feel you need to do so.



Noodle Needler: Make a nine-sided convex (all sides "bending out") polygon on your geoboard. Find its area and perimeter.

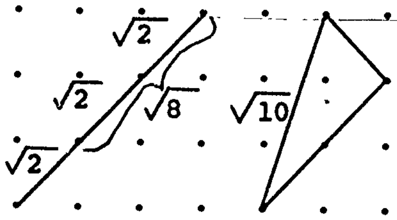
Answers are on page 84.

IX. We've found some new numbers! $\sqrt{2}$ has no counting number name. Can we add them? The previous exercise provided problems where we had to add them to themselves and to counting numbers. To make sure we've done it correctly, let's look at this example:



We can see that $\sqrt{8} = 2\sqrt{2}$.

On the geoboard, $\sqrt{8} + \sqrt{2} \neq \sqrt{10}$.



We find that the perimeter of our triangle has no name other than

$$\sqrt{2} + \sqrt{8} + \sqrt{10} \quad \underline{\text{or}}$$

$$3\sqrt{2} + \sqrt{10} \quad \underline{\text{or}}$$

$$\sqrt{2} + 2\sqrt{2} + \sqrt{10}$$

but no counting number name. These <u>irrational</u> numbers are clearly different!

X. Activities you might like to try now are:

(1) Working with the Pythagorean theorem, using 5 x 5 geoboards.

(2) Finding Pythagorean triplets or three numbers that can be sides of right triangles.

(3) Locating rational approximations of irrational numbers on the number line. See Pearcy and Lewis, <u>Experiments in Mathematics: Stage 2.</u>

(4) Learning to use square-root tables, successive approximations, slide rules, calculators, to find rational approximations of our strange new numbers.

(5) Pursuing the algebra of the irrationals. See Brumfiel, Eicholz, and Shanks, <u>Algebra I</u>.

(6) Finding all the different (in area) right triangles that can be made on a 5 x 5 geoboard, and finding their perimeters and areas.

<u>Answers to VI, VII, VIII</u>

<u>Answers to VI</u>

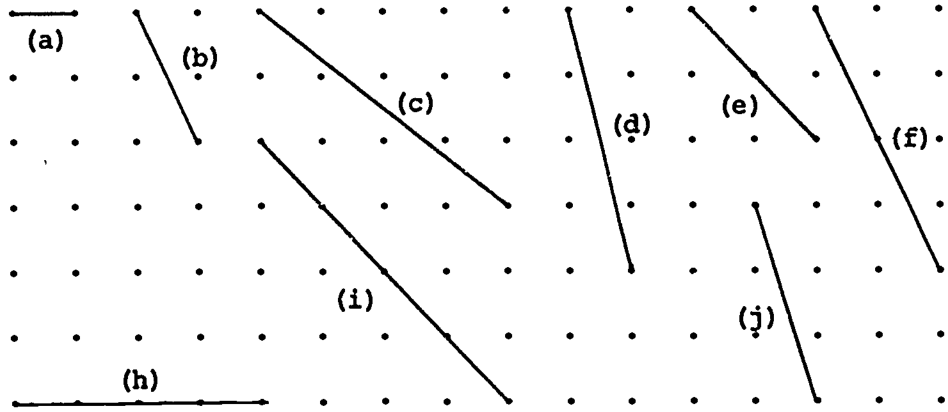
(a) 1 unit	(e) $\sqrt{5}$ units	(i) $\sqrt{17}$ units
(b) 3 units	(f) $\sqrt{5}$ units	(j) $2\sqrt{5}$ or $\sqrt{20}$
(c) 2 units	(g) $\sqrt{10}$ units	
(d) $\sqrt{2}$ units	(h) $\sqrt{8}$ or $2\sqrt{2}$ units	

Noodle Needler hints:

(1) There are more than 12.

(2) You can do it! Check your work with a friend; see if you agree.

Answers to VII

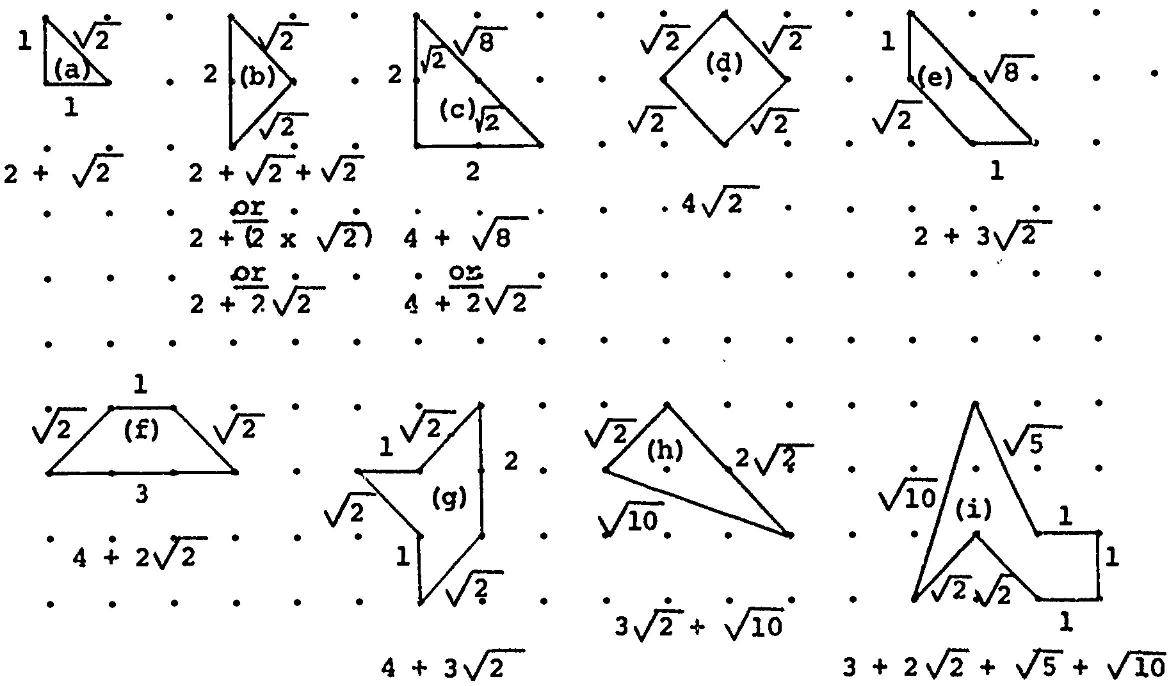


(g) is the same as (e). Can you prove this?

(k) is the same as (f). Can you prove this?

Noodle Needler: Check your work with a friend!

Answers to VIII



Noodle Needler: You're on your own!!!!

IV

ALGEBRA

There are two important things to remember from this section: that algebra can begin in first grade, and that the concept of a function is a central one to be explored at many grade levels.

A. Concepts and objectives

1. Variables, truth sets, open sentences; true, false, and open sentences; rule for substitution

2. Functions, Guess-My-Rule, finite differences

3. Graphing, number lines, Tic-Tac-Toe, designs, graphing linear equations, graphing Guess-My-Rule

4. Identities, shortening lists of identities, derivations (distinguishing between strategies and weapons)

5. Balances and balance pictures, equivalent equations

6. Nonlinear graphing

7. Open sentences with more than one variable

8. Lattices

9. Matrices

B. Materials

Dienes algebraic blocks and multibase blocks

Madison Project shoebox puzzles

Cuisenaire rods, squares, and cubes

Balance beams

C. Resources (see Bibliography for complete references)

Davis, Discovery in Mathematics, Explorations in Mathematics (teacher's and student's editions of both)

Nuffield Mathematics Project booklet, Graphs Leading to Algebra

Page, Maneuvers on Lattices, Number Lines, Functions and Fundamental Topics

Sawyer, The Search for Pattern

Informal Algebra

Many classrooms are beginning an informal approach to algebra in the early grades. Some excellent materials are available. We would particularly recommend two volumes by Robert B. Davis: Discovery in Mathematics, and Explorations in Mathematics, both published in 1963 by Addison-Wesley.

In both cases we would recommend the purchase of the teacher's edition. The introductions, particularly that in Discovery, are excellent.

Discovery was written for the middle grades, but it certainly can be easily adapted for the primary level. Explorations is intended for use at the junior high level, but can be easily adapted for elementary grades or for secondary schools.

Among the topics covered in depth, and with great understanding, are those that are at the core of every formal course in algebra: (a) variables and open sentences, (b) functions approached through Guess-My-Rule games, (c) equations and inequalities, (d) graphing, (e) identities and derivations, (f) lattices, (g) matrices, and (h) word problems.

Some of these topics may sound a bit esoteric for mathematics below high school. Yet how often do we underestimate the power of our students or avoid topics because we've never studied them ourselves?

Try it; you'll like it.

* * * * * * *

Following are several worksheets on true, false, and open sentences, including quadratics; number lines; Tic-Tac-Toe; squares; Guess-My-Rule, and linear equations and functions. The teacher can make up many more of his own, and the children will both enjoy and profit from making up others for their classmates. Valuable suggestions will be found in Davis, Discovery and Explorations, and in Page, Number Lines, Functions and Fundamental Topics.

True, False, and Open Sentences, and Rule for Substitution: Worksheet

1. Given the following sentences, tell which are true, which false, or which open.

(a) $3 + 4 = 7$ ________________

(b) $6 + 2 > 5$ ________________

(c) $3 + 4 < \square$ ________________

(d) $5 + 1 = 7$ ________________

(e) $7 + 1 \neq \triangle$ ________________

(f) $-3 + 4 < 0$ ________________

(g) $5 - 6$ ________________

2. Below are three columns. In column I, you will find an open sentence. In column II, you will find a substitution that has been made in these open sentences. In column III, state whether this substitution is legal or illegal.

I	II	III
(a) $\square + 1 = \square \times 1$	$\boxed{8} + 1 = \boxed{9} \times 1$	
(b) $-4 + \triangle = -3$	$-4 + \triangle{+1} = -3$	
(c) $\triangle + \square = 4$	$\triangle{2} + \boxed{2} = 4$	
(d) $\nabla + \triangle = -2$	$\nabla{-3} + \triangle{1} = -2$	
(e) $\square + \nabla = \triangle$	$\boxed{-2} + \nabla{-2} = \triangle{4}$	
(f) $\square + \square = 8$	$\boxed{5} + \boxed{3} = 8$	

3. For the open sentences below, make the indicated substitutions.

(a) $\square + \square = 7$ Make an illegal and false substitution. (Put your answer in the frames.)

(b) $\square + \square = 7$ Make an illegal and true substitution.

(c) $\square + \square = 7$ Legal and false substitution.

(d) $\square + \square = 7$ Legal and true substitution.

(e) $\square + \triangle = 8$ Legal and true substitution.

(f) $\square + \triangle = 8$ List three more legal and true substitutions.

(g) $\square + \triangle = 8$ Legal and false substitution.

(h) Can you make an illegal and false substitution for

$\square + \triangle = 8$?

True, False, and Open Sentences, and Rule for Substitution: Answer Sheet

1. (a) T
   (b) T
   (c) Open
   (d) F
   (e) Open
   (f) F
   (g) not a sentence

2. (a) illegal
   (b) legal
   (c) legal
   (d) legal
   (e) legal
   (f) illegal

3. Note: These are only examples. There are other possibilities (except for letters d and h).

(a) $\boxed{3} + \boxed{5} = 7$

(b) $\boxed{3} + \boxed{4} = 7$

(c) $\boxed{4\frac{1}{2}} + \boxed{4\frac{1}{2}} = 7$

(d) $\boxed{3\frac{1}{2}} + \boxed{3\frac{1}{2}} = 7$

(e) $\boxed{3} + \triangle\!\!\!\!\!\:5 = 8$

(f) $\boxed{5} + \triangle 3 = 8$, $\boxed{6} + \triangle 2 = 8$, $\boxed{7} + \triangle 1 = 8$

(g) $\boxed{6} + \triangle 3 = 8$

(h) No.

True, False, and Open Sentences: Worksheet

Which are true? Which are false? Find numbers that make the open sentences true.

1. $3 + \square = 5$
2. $2 > 6$
3. $\square \times 2 = 6$
4. $4 + (2 \times 1) = 6$
5. $\square + \square = 6$
6. $1 + (3 \times \square) = 13$
7. $\square - \square = 0$
8. $\square \div 2 = 8$
9. $\square + \square = 2 \times \square$
10. $\square \times \square = 9$
11. $4 < \square < 8$
12. $2 \times \square < 16$
13. $\square / \square = 1$
14. $\square \times \square \times \square = 8$
15. $\square + \triangle = \triangle + \square$
16. $\square + \triangle = 12$
17. $(3 \times \square) + 4 = 10$
18. $1/\square + 1/\square = 1$
19. $\square - \square = 1$
20. $\square + \triangle = 1$
21. $3 > \square > 2$
22. $\square / \triangle = 1/2$

True, False, and Open Sentences: Answer Sheet

1. 2
2. False
3. 3
4. True
5. 3
6. 4
7. Any number
8. 16
9. Any number
10. 3 or -3
11. Any number between 4 and 8
12. Any number less than 8
13. Any number but zero
14. 2
15. $\square$ = any number

    $\triangle$ = any number
16. Any pair whose sum is 12
17. 2
18. 2
19. No number will make this true.
20. Any pair whose sum is 1
21. Any number between 2 and 3
22. Any pair in which $\square$ is half of $\triangle$

More Open Sentences: Worksheet

Find the numbers that make these open sentences true.

1. $5 + (2 \times \square) = 25$
2. $3 + (2 \times \square) = 203$
3. $3 + (2 \times \square) = 8$
4. $7 + (2 \times \square) = 109$
5. $1 + (2 \times \square) = 102$
6. $5 + (3 \times \square) = 18$
7. $4 + (2 \times \square) = 7$
8. $(2 \times \square) - 4 = 10$
9. $(2 \times \square) + 7 = 28$
10. $(3 \times \square) + 2 = 9$
11. $(3 \times \square) + 1 = (1 \times \square) + 21$
12. $(5 \times \square) + 3 = (4 \times \square) + 9$
13. $(5 \times \square) + 3 = 24$
14. $(3 \times \square) + 10 = 15$
15. $(3 \times \square) + 5 = 13$
16. $(7 \times \square) + 1 = 39$
17. $(5 \times \square) + 10 = 61$
18. $(2 \times \square) + 193 = 199$
19. $(3 \times \square) + 2 = (2 \times \square) + (1 \times \square)$
20. $(\square \times \square) - (9 \times \square) + 14 = 0$

More Open Sentences: Answer Sheet

1. 10
2. 100
3. 2 1/2
4. 51
5. 50 1/2
6. 13/3
7. 3/2
8. 7
9. 21/2
10. 7/3
11. 10
12. 6
13. 21/5
14. 5/3
15. 8/3
16. 38/7
17. 51/5
18. 3
19. No number
20. 7 or 2

Even More Open Sentences: Worksheet

Find a number that will make these open sentences true. Find more than one if you can.

1. $(2 \times 6) + 2\ 1/2 = \square$

2. $3 - (2 \times \square) = 4$

3. $15 - 2 \times (\square + 6) < 10$

4. $0 \times \square = 0$

5. $3 < \square < 3$

6. $(8 \times \square) + 5 = 73$

7. $5 \times \square = -15$

8. $-4 + (\square \times \square) = 5$

9. $3 \times (\square + \triangle) = (3 \times \square) + (3 \times \triangle)$

10. $156\ 5/9 \times \triangle = \triangle \times 156\ 5/9$

11. $(100 \times \square) - (25 \times \square) = \triangle \times 75$

12. $\square + (5 + 11) = (\square + 5) + 11$

13. $10/3 \times \square = 25$

14. $1/\square \times \square = 1$

15. $6.3 \times \square = -.441$

16. $(\square \times \square) + 1/2 = 2/3 + 5/6$

17. $(2 \times 5) \div \square < 0$

18. $6 \times \square = 6$

19. $\square \times \triangle = 0$

20. $4 \times \square = (3 \times \square) + (1 \times \square)$

Even More Open Sentences: Answer Sheet

1. 14 1/2
2. -1/2
3. Any number greater than -3 1/2
4. Any number
5. No number will make this true.
6. 8 1/2
7. -3
8. 3 or -3
9. Any pair of numbers
10. Any number
11. Any pair of equal numbers
12. Any number
13. 7 1/2
14. Any number but zero
15. -.07
16. 1 or -1
17. Any negative number
18. 1
19. Any pair in which at least one number is zero
20. Any number

Quadratics: Worksheet

(There are two numbers that will make these true. Look for some secrets.)

1. $(\square \times \square) - (5 \times \square) + 6 = 0$

2. $(\square \times \square) - (8 \times \square) + 15 = 0$

3. $(\square \times \square) - (15 \times \square) + 50 = 0$

4. $(\square \times \square) - (13 \times \square) + 22 = 0$

5. $(\square \times \square) - (102 \times \square) + 200 = 0$

6. $(\square \times \square) - (17 \times \square) + 70 = 0$

7. $(\square \times \square) - (37 \times \square) + 70 = 0$

8. $(\square \times \square) - (108 \times \square) + 800 = 0$

9. $(\square \times \square) - (7 \times \square) + 10 = 0$

10. $(\square \times \square) - (28 \times \square) + 75 = 0$

11. $(\square \times \square) - (16 \times \square) + 55 = 0$

12. $(\square \times \square) - (107 \times \square) + 700 = 0$

13. $(\square \times \square) - (20 \times \square) + 96 = 0$

14. $(\square \times \square) - (12 \times \square) + 11 = 0$

15. $(\square \times \square) - (11 \times \square) + 30 = 0$

16. $(\square \times \square) - (13 \times \square) + 30 = 0$

17. $(\square \times \square) - (29 \times \square) + 100 = 0$

18. $(\square \times \square) - (15 \times \square) + 26 = 0$

19. $(\square \times \square) - (53 \times \square) + 150 = 0$

20. $(\square \times \square) - (23 \times \square) + 42 = 0$

21. $(\square \times \square) - (4 \times \square) + 3 = 0$

22. $(\square \times \square) - (104 \times \square) + 303 = 0$

23. $(\square \times \square) - (12 \times \square) + 35 = 0$

24. $(\square \times \square) - (34 \times \square) + 93 = 0$

25. $(\square \times \square) - (30 \times \square) + 189 = 0$

Quadratics: Answer Sheet

1. 3 or 2
2. 5 or 3
3. 10 or 5
4. 11 or 2
5. 100 or 2
6. 10 or 7
7. 35 or 2
8. 100 or 8
9. 5 or 2
10. 25 or 3
11. 11 or 5
12. 100 or 7
13. 8 or 12
14. 11 or 1
15. 6 or 5
16. 10 or 3
17. 25 or 4
18. 13 or 2
19. 50 or 3
20. 21 or 2
21. 3 or 1
22. 101 or 3
23. 7 or 5
24. 31 or 3
25. 21 or 9

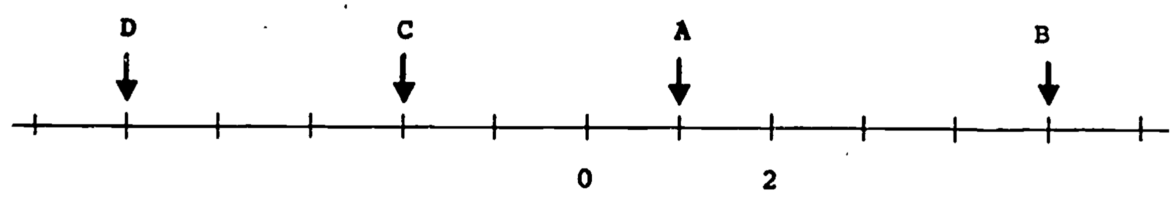
The conventional way to write the responses to these is $\{3,2\}$. This means that if we put 3 in all of the boxes, the sentence will be true, <u>and</u> if we put 2 in all of the boxes, the sentence will be true. Thus 3 will work; 2 will work.

This notation is different from (3,2), in parentheses, which would be one acceptable solution for $\square + \triangle = 5$, where, if we put 3 in the box and 2 in the triangle, we would get a true sentence.

Number Lines, Crossed Number Lines, and Tic-Tac-Toe: Worksheet

In figures 1 and 2, name the places indicated by arrows A, B, C, D, E, F, G, H, and I.

1.



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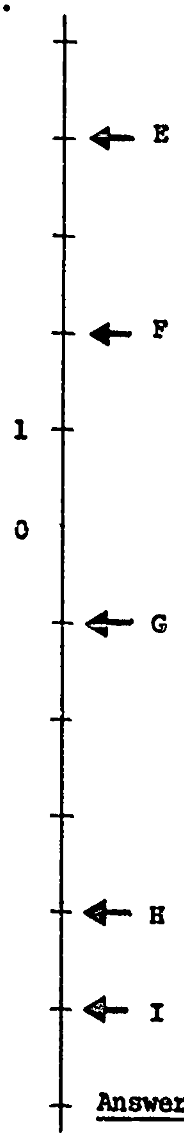

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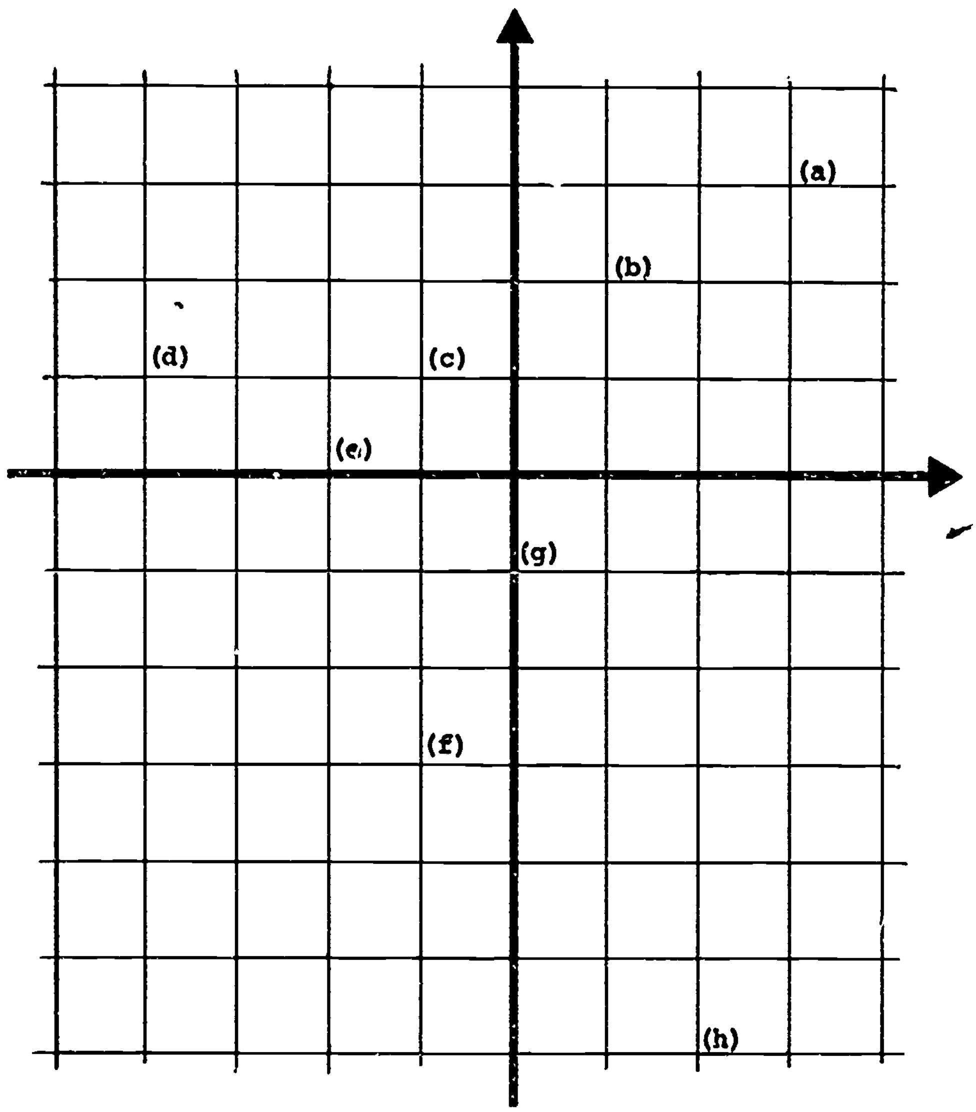


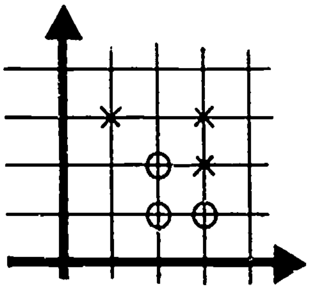
Answers:

A. ______ F. ______
B. ______ G. ______
C. ______ H. ______
D. ______ I. ______
E. ______

In figure 3, can you name intersections (a), (b), (c), (d), (e), (f), (g), and (h)?

(a) ______ (e) ______
(b) ______ (f) ______
(c) ______ (g) ______
(d) ______ (h) ______

4.



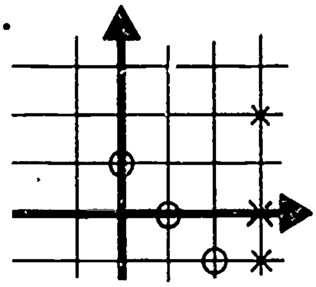
(It takes four in a row to win)

It is X team's turn.

Where should the next X be place to assure winning the game?

(,)

5.

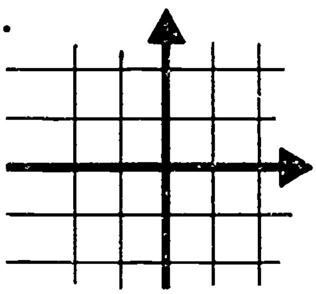


It is O team's turn.

Where would you place the next O?

(,)

6.



O team:	(0,0)
X team:	(2,1)
O team:	(2,2)
X team:	(0,1)
O team:	(1,1)
X team:	(1,0)

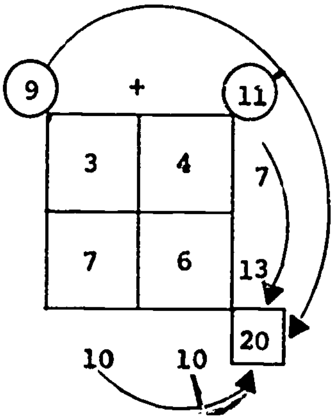
After the above moves, team O needs (,) to win. If team O chooses (1,2), name two points needed by X team to win:

(,) and (,)

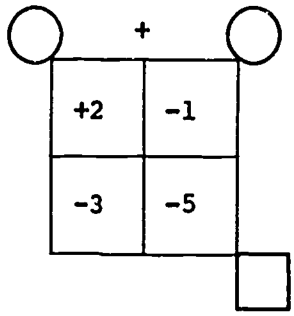
How about adding other varieties of this game, where you not only name the grid reference but, for example, must also multiply the two numbers. You could use a segment of the grid with larger numbers to give practice with X^n facts. Many possibilities with this.

Mickey Mouse Squares: Worksheet

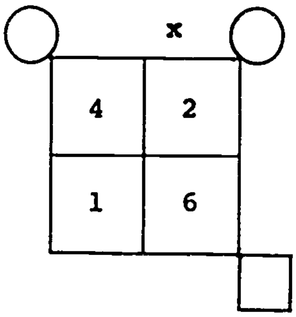
1.



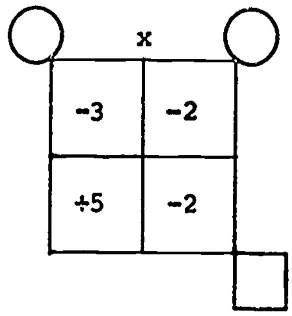
2.



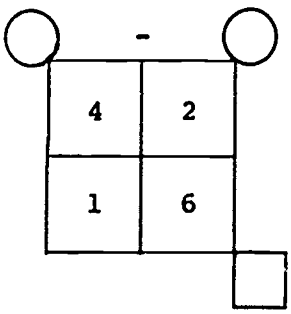
3.



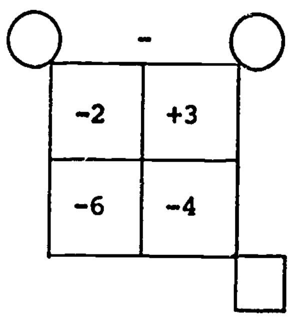
4.



5.



6.



For further information on these, see Wirtz et al., Math Workshop: Games and Enrichment Activities.

Guess-My-Rule: Worksheet

1. In playing Guess-My-Rule, Roy made the following table:

□	△
7	17
4	11
10	23
.	.
.	.

What was his rule?

Put some more numbers in the table.

2. Later on, Frank made this table:

□	△
2	1
5	16
3	6
1	-4
0	-9
4	.
6	.
-2	.
.	.
.	.
.	.

What was his rule?

Complete the table.

3. In another game, children found that some numbers were more revealing than others. New table:

□	△
0	-7
10	33
100	393
7	21
3	5
.	.
.	.

What is this rule?

4. After games 2 and 3, above, they thought that they were on to some pattern, but they were torpedoed with the following:

$\square$	$\triangle$
0	3
10	103
100	10,003
4	19
7	52
.	.
.	.

What is this clever rule?

5. Rule 5 (for experts:

$\square$	$\triangle$
0	2
4	130
5	252
10	2002

What is this rule?

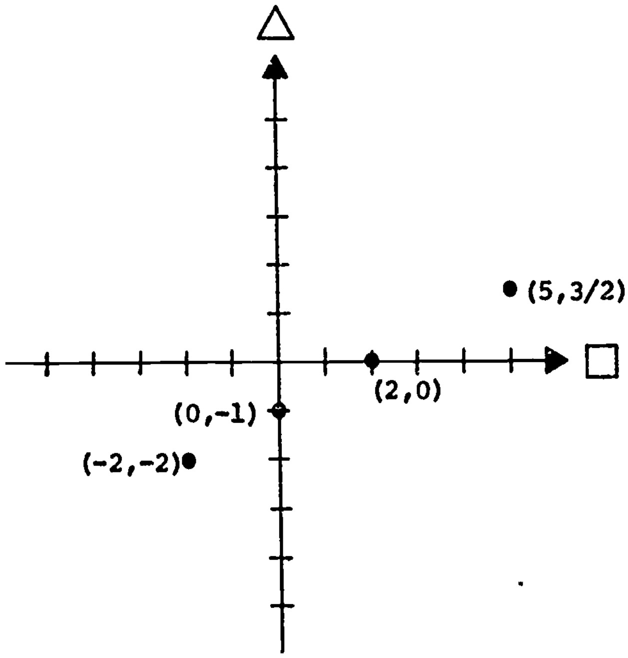
6. Given the ordered pairs (0,2), (1,3), 2,4), and (-1,1), tell:

a. The second coordinate of the ordered pair (4,___).

b. The first coordinate of the ordered pair (___,-1).

c. The above ordered pairs are elements of a linear function. Give a rule defining this linear equation.

7. The following points are in the graph of a linear equation. Write an open sentence defining this linear equation (see next page):



8. Two ordered pairs in the truth set of a linear equation are (0,-2) and (1,-5). Write this equation.

9. The open sentence is (-2 x □) + 3 = △

If □ = -2 and a child maintains that -2 x -2 = -4, what would you use to help him out?

a. Pebbles-in-the-Bag?

b. Postman Stories?

c. Multiplying two negatives always gives a positive product?

d. Pet Shop Stories?

Guess-My-Rule: Answer Sheet

1. $(2 \times \square) + 3 = \triangle$

2. $(5 \times \square) + -9 = \triangle$ or $(5 \times \square) - 9 = \triangle$

3. $(4 \times \square) + -7 = \triangle$ or $(4 \times \square) - 7 = \triangle$

4. $(\square \times \square) + 3 = \triangle$

5. $2\square^3 + 2 = \triangle$

6. (a) (4,6)

   (b) (-3,-1)

   (c) $\square + 2 = \triangle$

7. $(1/2 \times \square) + -1 = \triangle$

8. $(-3 \times \square) + -2 = \triangle$

9. (b)

Graphing Open Sentences: Worksheet

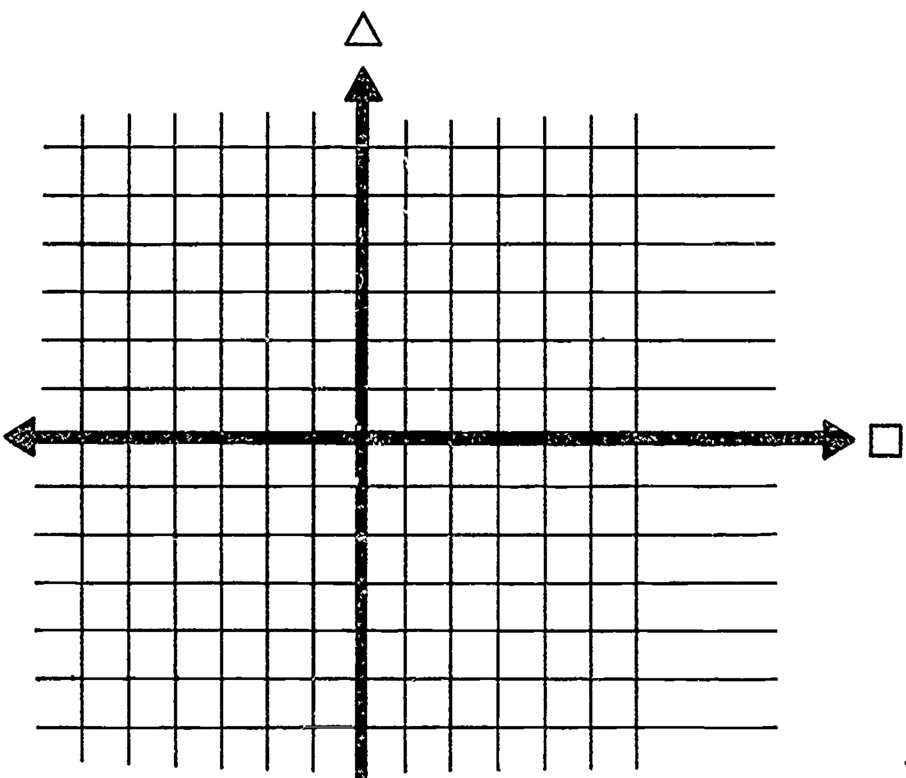
1. Can you finish the pairs in the truth set for this open sentence?

$\square + 9 = \triangle$

$\square$	$\triangle$
4	
0	
-2	
-3	
	10
	-3

2. Graph the following set of ordered pairs:

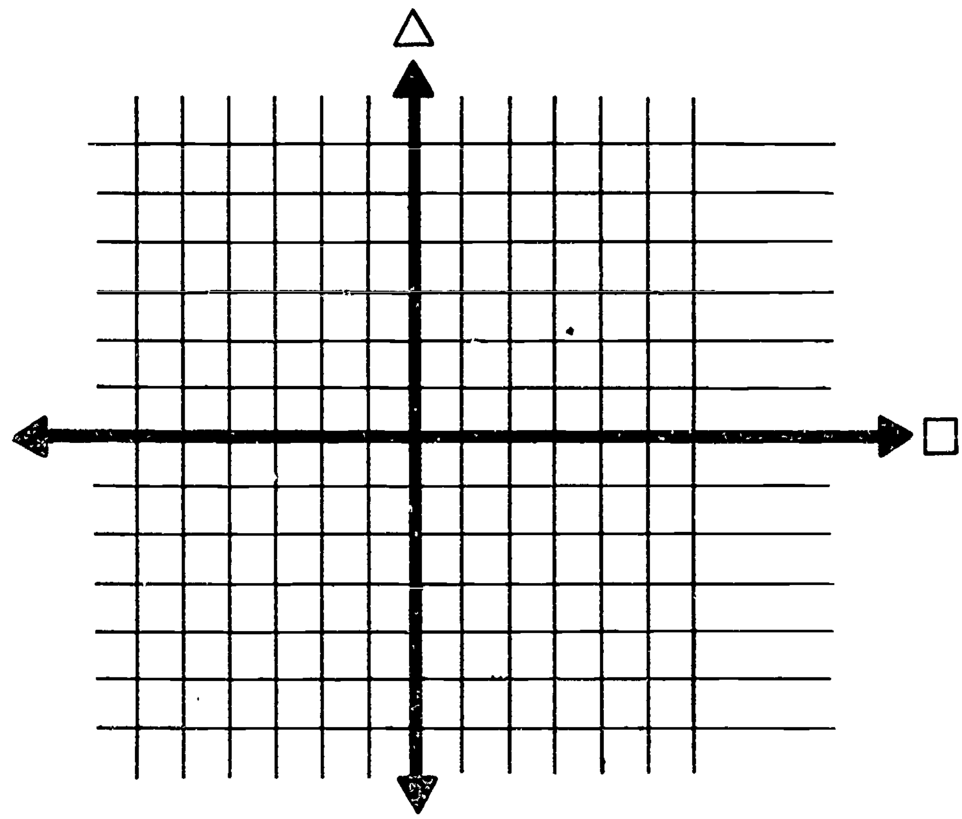
(1,2), (0,0), (-1,-2), (-2,-5), (2,5)



3. Graph the truth set of the open sentence (-2 x □) + -1 = △.

The replacement set for □ is the set {-2, -1, 0, 1, 2}.

□	△

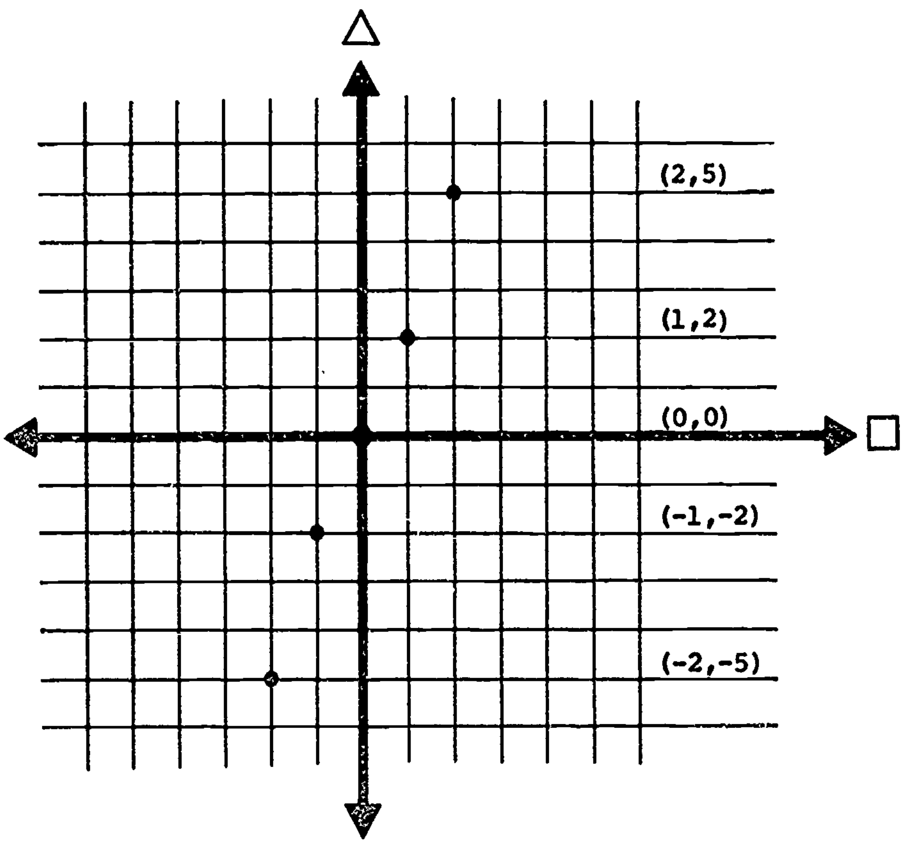


Graphing Open Sentences: Answer Sheet

1.

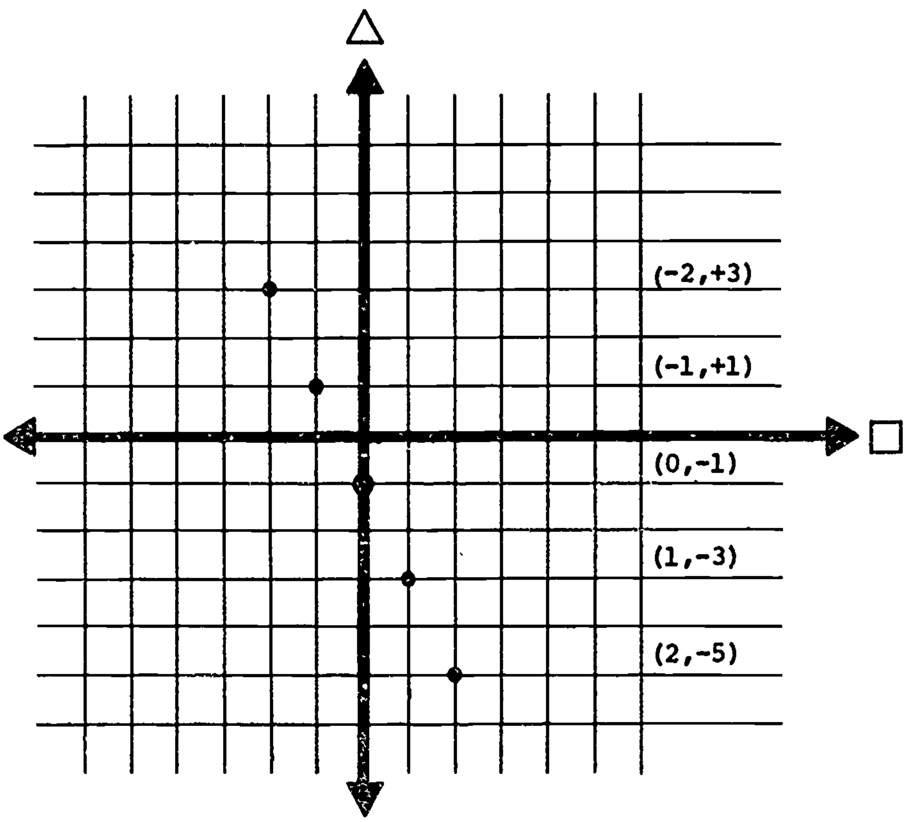
□	△
4	13
0	9
-2	7
-3	6
1	10
-12	-3

2.



3.

□	△
-2	3
-1	1
0	-1
1	-3
2	-5



V

INFORMAL GEOMETRY

A. Concepts and objectives

1. Shape: two- and three-dimensional

2. Space and size

3. Symmetry

4. Congruence, similarity: informal activities where "same" and "different" are deliberately not well-defined so that children will arrive at a notion of congruence and of similarity. Many activities with a geoboard.

5. Parallelism

6. Points, lines, planes, space

7. Motion geometry: slides, turns, and flips

8. Special relations with angles

9. Geometry with a circular geoboard

10. Rubber-sheet geometry (topology)

11. Faces, edges, vertices

12. Constructions

13. Trigonometry approached through similar triangles

B. Resources (see Bibliography for complete references)

Altair Designs (game)

Bell, Mathematics in the Making (see especially booklets entitled Curves, Looking at Solids, Rotation and Angles, and Transformations and Symmetry)

Cohen, Inquiry in Math via Geoboards

Gattegno, Geoboard Geometry

Nuffield Mathematics Project booklet, Environmental Geometry

Rowland, Looking and Seeing

Snow Crystals (game)

Trivett, Geoboard Activity Cards, Introducing Geoboards

University of Illinois Committee on School Mathematics, Motion Geometry

Walter, Boxes, Squares, and Other Things (unit of informal geometry initiated with half-pint milk cartons and scissors)

SPACE, SIZE, SHAPE, SYMMETRY

Materials

Poleidablocs

Mosaics

Jigsaw puzzles

Different-size balls, wheels, hoops, tin lids that can be graded

Basic shapes set

Geostrips

Large and small cubes, long blocks, flat blocks, cones, arches, planks, for building

Geoblocks

All kinds of boxes and materials for building scale models

Three-dimensional shapes: cuboids, pyramids, prisms, spheres, etc.

Tins, boxes, bottles of all shapes

Interesting shapes in nature: pieces of wood, leaves, rocks, minerals, crystals

Kaleidoscopes

Mirrors

Mirror cards

Wax and tracing paper for folding

Squared paper, colored paper, etc.

Objectives

Concept of size, shape: squares, rectangles, trapezoids, parallelograms, triangles, rhomboids, cones, cylinders

Use of shape in the environment

Simple architecture (floor plans and 3-D models)

Symmetries: flip, slide, rotation

Similar figures: ratio and proportion

VI

SETS, LOGIC, AND PROOF

A. Concepts and objectives

1. Sets (junk, attribute blocks, people pieces)
2. Complement, subset
3. Ordering, classifying, introducing notation
4. And, or, not, implication
5. Tautologies, contradictions
6. Making lists of identities
7. Shortening lists--"super identities"
8. Use of variable
9. Principle of names
10. Derivations

Special unit on logic and proof included in this section.

B. Materials

Collections of all kinds of things for sorting and classifying (see Section I, page 13, for a detailed list)

Logic blocks

Attribute blocks

People pieces

Creature cards

C. Resources (see Bibliography for complete references)

Charbonneau, Logic with Hidden Rods

Davis, Discovery in Mathematics, Explorations in Mathematics

Hull, Attribute Games and Problems

Logic and Proof

Most of the material presented in this notebook is related to mathematical experiences that are at the core of the curriculum presented in all of our classrooms. The unit that follows is definitely an interesting extra. Presentations similar to this have been found to be pleasurable and beneficial in the classrooms in which they have been tried. We plan to submit other interesting departures in subsequent issues of this notebook.

This unit assumes that the children are familiar with logic blocks and that they have had experience in sorting, ordering, and grouping. If not, some work should be done in these areas.

Simple versus compound sentences

A simple sentence expresses a single idea. Taking any logic block (for example, the small, thin, blue triangle), one can say, "This piece is a triangle," "This piece is blue," "This piece is thin," or "This piece is small."

To lead into the discussion of compound sentences, the teacher might ask if the statement "This piece is a blue triangle" is a simple sentence. To emphasize the fact that this statement contains two ideas, have the students break it down into two simple sentences connected by "and," thus forming a compound sentence: "This piece is blue and this piece is a triangle."

To investigate "and," (a) use other single blocks, as above; (b) use two blocks identical in two attributes (for example, "These pieces are blue." "These pieces are blue and they are large."); (c) use three blocks identical in all but one attribute (for example, "These pieces are thin." "These pieces are small and these pieces are blue.").

The children will readily see that, if the word "and" is used, both attributes mentioned must be true. Try alternate wordings: "This piece is both thin and blue," for example.

A second type of compound sentence uses the word "or." Again using the small, thin, blue triangle, one might ask if it is correct to say: "This piece is blue or this piece is thin." "This piece is a triangle or this piece is small." "This piece is blue or this piece is red. "This piece is a square or this piece is thin." After everybody has agreed that all of the statements above are correct, the teacher can ask the children how the first two statements differ from the last two to motivate discussion of the inclusive "or" (both attributes present) versus the exclusive "or" (one or the other of the attributes true, but not both).

To investigate "or," (a) use other single blocks, as above; (b) use various sets of the blocks that differ in one attribute only; (c) use various sets that differ in two or more attributes. Be sure that everyone understands that a compound sentence is always made up of two simple sentences. If a child speaks of a "small, thin block," be sure he knows which two simple sentences are involved.

In the preceding work, someone is likely to introduce the idea of negation. Although it seems to express a single idea, the simplest negation is a compound sentence. For example, using our set of blocks, "This piece is not red" implies that "This piece is blue or it is yellow."

To investigate "not," (a) use single blocks; (b) use various sets composed of like pieces and ask the students to describe the remaining set. (For example, have a child choose all the blocks with a common attribute, say, red. Then the remaining set can be described as "not red.")

To help with the written expression of these ideas, some sort of notation should be introduced. The children can be led to suggest something like this:

L	large	R	red	$\wedge$	and
S	small	Y	yellow	$\vee$	or
K	thick	B	blue	$\sim$	not
H	thin				

Conventional geometric symbols may be used for shape. To emphasize the structure of compound sentences, parentheses are often used, especially if the sentence involves one or more compound sentences; for example, $(L \wedge R) \vee (B \wedge K)$.

Problems with Logic Blocks

1. Make up sets of blocks to fit the following statements:

(a) $R \wedge \odot$

(b) $S \vee B$

(c) $\sim B$

(d) $R \vee Y$

(e) $\sim (Y \vee B)$

(f) R

(g) $\sim B \wedge \sim R$

(h) Y

(i) $(R \wedge K) \wedge S$

(j) $R \wedge (\triangle \vee \square)$

(k) $(R \wedge \odot) \vee (B \wedge L)$

(l) $(S \wedge K) \wedge \sim R$

Similar exercises using discrete sets of numbers would be useful. For example:

Choose a set of numbers satisfying each of these sentences:

(a) These numbers can be divided by 5.

(b) These numbers can be divided by 5 and are odd numbers.

(c) These numbers can be divided by 5 or are odd numbers.

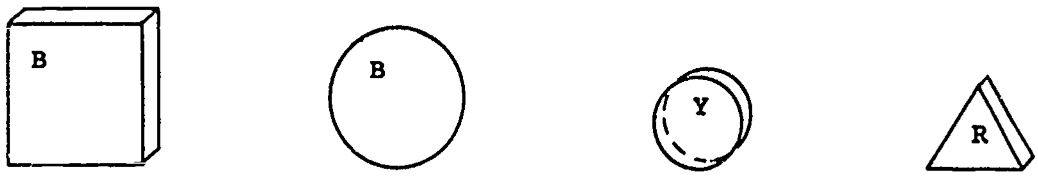
(d) These numbers cannot be divided by 5. Etc.

The possibilities for illustrating "and" and "or" are endless. For example, a class could be given attributes like sex, hair color, eye color, and the like, to illustrate compound sentences such as "Blue-eyed boys or brown-eyed girls," etc.

Conditional sentences

One of the most important ideas in logic is the concept of the conditional, usually expressed "If _____, then _____." If a simple sentence is symbolized by a letter, this might appear as "If R, then K," or, more simply still, "R ⟶ K." At this point it would be well to remember the excellent advice of W. W. Sawyer, in his book The Search for Pattern: "Terminology should not be taken too seriously. It is characteristic of poor teachers that they lay great stress on words and little stress on ideas. . . . Effective thinking in mathematics depends not on remembering words but on understanding situations."

An understanding of the conditional is easily motivated by the use of logic blocks. For example, the teacher holds up four blocks, such as these,

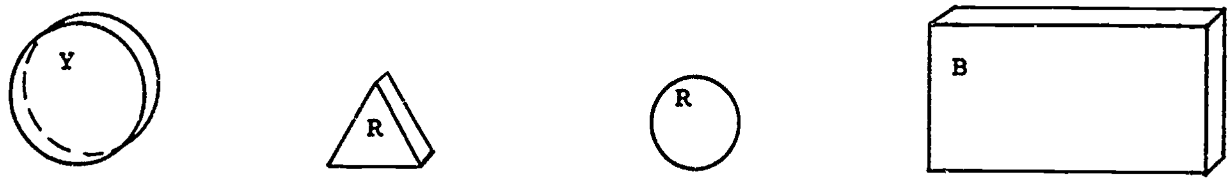


(large, thick) (large, thin) (small, thick) (small, thick)

and says, "How can I finish this sentence--'If blue, then _____'?" Using the same blocks, she might ask the children to think of other true sentences starting with "If" and get such things as "If a square, then blue," "If thin, then a circle," etc. She will also get more complex conditionals, such as "If yellow or red, then small." She will accept these as interesting, but probably will have better luck if she directs attention to simple conditionals. After sufficient practice, she may say, "Let's interchange the 'if' and 'then' parts of our sentences." "If a square, then blue" becomes "If blue, then a square." "If thin, then a circle" becomes "If a circle, then thin." "If a triangle, then red" becomes "If red, then a triangle." And so on. The children will notice quickly that some of these turned-about (converse) sentences are true and some are not. The teacher may then ask that the class find as many true sentences as possible that have correct converses (she doesn't have to use the term "converse" unless she wishes). This will motivate the idea of the biconditional, or equivalence; for example, "If red, then a triangle, and if a triangle, then red," symbolized as $R \leftrightarrow \triangle$.

Problems with Conditional Statements

Choose a set of blocks:



(large, thick) (small, thick) (small, thin) (large, thick)

1. Finish each conditional with a true simple sentence:

   (a) If yellow, then ____.

   (b) If red, then ____.

   (c) B $\longrightarrow$ ____.

   (d) H $\longrightarrow$ ____.

   (e) ____ $\longrightarrow \bigcirc$.

   (f) ____ $\longrightarrow \square$.

2. (a) Write the converse of each of the conditionals in problem 1.

   (b) Which converses are true?

   (c) Write each of the sentences in problem 1 that has a true converse as a biconditional.

3. Using logic blocks, find a set to illustrate these sentences:

   (a) $H \longrightarrow R$

   (b) $B \longrightarrow (\triangle \wedge \square)$

   (c) $(L \vee S) \longrightarrow B$

   (d) $L \vee (S \longrightarrow B)$

   (e) $Y \longleftrightarrow S$

   (f) $(R \vee Y \vee B) \longrightarrow H$

Truth tables

It is quite easy to motivate truth tables for conjunction, disjunction, negation, conditional, and biconditional by the use of logic blocks. After this, the truth values of much more complex sentences may be investigated. The time available and the ingenuity of the teacher are the only limitations.

To motivate the truth table for conjunction, pick one block, say, the large, red, thick, square piece. Have the students produce conjunctions: red and large, red and small, blue and large, blue and small, thick and square, etc. These might be tabulated like this:

Color	Size	Value
R	L	True (T)
R	S	False (F)
B	L	F
B	S	F
		etc.

or

Size	Shape	Value
L		T
L		F
S		F
S		F
		etc.

The students will see that, in order for a conjunction to be true, both simple sentences involved must be true, and they will accept the table

A	B	$A \wedge B$
T	T	T
T	F	F
T	F	F
F	T	F
F	F	F

where A and B represent simple sentences, and T and F the truth or falsity of the sentences.

Since they have already observed that a disjunction is true if one or both simple sentences involved are true, children will develop the table for disjunction without fuss. If not, again motivate it, using a single block:

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Negation presents no problem. If the block is red, then it is false that it is anything else. This might be a good time to speak of set complements; the complement of the set of red blocks is the set of anything but red, in this case yellow or blue. This might be discussed if the students wants to say something like "If it is not red, then it is blue," which of course is not necessarily so.

A	$\sim A$
T	F
F	T

The idea of the truth table for the conditional presents a little more difficulty. Using the same large, thick, red, square piece, the child will agree that "If red, then thick" is true and that "If red, then thin" is false; it may be hard to get him to accept "If blue, then thick" and "If blue, then thin" as both being true. Point out that these statements only say what will happen if the block is blue. Since it is red, not blue, neither sentence can be disproved and hence is accepted as true. Therefore we have the table

A	B	$A \longrightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

At this point, an excellent way to extend the concept of truth tables is to develop a table for the biconditional. Remind the children that they know that the biconditional is a compound sentence stating that $A \longrightarrow B$ and $B \longrightarrow A$, which is written $A \longleftrightarrow B$. Therefore,

A	B	$A \rightarrow B$	$B \rightarrow A$	$(A \rightarrow B) \wedge (B \rightarrow A)$	$A \leftrightarrow B$
T	T	T	T	T	F
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Using the basic truth tables, it is now possible to investigate the validity of any number of complicated sentences, such as $R \vee \sim(R \wedge K)$. To illustrate:

R	K	$R \wedge K$	$\sim(R \wedge K)$	$R \vee \sim(R \wedge K)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

This sentence, $R \vee \sim(R \wedge K)$, is a tautology, which means that it remains true no matter what the truth values of the sentences that make it up. Students will enjoy picking blocks to illustrate a tautology. For example, for the first line they might pick all thick, red blocks; for the second, all thin, red blocks; then all blue or yellow thick blocks; then all blue or yellow thin blocks. In each case, the main sentence will be true.

The children will be interested in investigating other compound sentences and in illustrating their truth values with the blocks. Here are some to try. If they are tautologies, they will be true no matter what combination of blocks is used.

(1) $B \leftrightarrow H$

(2) $R \leftrightarrow (R \vee R)$

(3) $Y \wedge (L \vee Y) \rightarrow L$

(4) $(R \wedge K) \rightarrow (R \vee K)$

Let the students make up compound sentences, pick a set of blocks, and investigate truth values. Excellent tautologies to investigate are the standard laws of inference, such as

(1) $(P \wedge (P \rightarrow Q)) \rightarrow Q$

(2) $((P \vee Q) \wedge \sim P) \rightarrow Q$

(3) $((P \rightarrow Q) \wedge \sim Q) \rightarrow \sim P$

(4) $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$

VII

MATHEMATICAL PATTERNS

A. Concepts and objectives

1. Odds and evens
2. Sequences
3. Triangular numbers
4. Square numbers
5. Rectangular numbers
6. Doubling
7. Halving
8. Number lines with sequences of all kinds
9. Signed numbers
10. Primes and composites
11. Prime factorization
12. Factors and multiples
13. Euclidian algorithm
14. Least common multiples
15. Greatest common factors
16. Game of Lee
17. Fun numbers

In short, to make children aware of the pattern and structure found throughout mathematics, to appreciate the beauty of its subtleties, and, JUST INCIDENTALLY, to improve the computation and skills in the process!

B. Materials

Cuisenaire rods

Cubes (as in Dienes blocks)

Color cubes (inch blocks)

Tiles

Marbles, pennies, etc.

Geoboards

Simple calculators

Paper of all kinds: graph in different sizes, construction, sticky, gummed, plain

Colored pencils, pens, felt-tip pens (fine and broad), crayons, paints

Anything you can think of that works, but remember the importance of COLOR to make patterns pop out!

Altair designs

Pattern blocks and pattern-block stickers

C. Resources (see Bibliography for complete references)

Davidson, Jessica, Using the Cuisenaire Rods

Fitzgerald et al., Laboratory Manual for Elementary Mathematics (see especially sections on primes and composites, prime factorization, factors and multiples, and chapter 6 on Euclidian algorithm)

Nuffield Mathematics Project booklets

Page, Number Lines, Functions and Fundamental Topics

University of Illinois Committee on School Mathematics, Stretchers and Shrinkers

PATTERNS

Most children are intrigued by the discovery of pattern in their work with numbers, especially when the same pattern is generated by seemingly unrelated problems. Almost as soon as a child can count he can make investigations that lead to exciting results of this sort, and the more arithmetic he can use the more he can find. Some children are content simply to observe the results, while others insist on pursuing matters in the effort (often successful) to find out why. Certainly the computational skills will grow in the process.

In general, the materials needed are very simple--

lots of paper (plain and graph)

pencils of all colors

Cuisenaire rods

cubes (as in Dienes blocks)

marbles, pennies, etc.

geoboards

simple calculators.

ODDS AND EVENS

A. Young children need lots of practice in identifying odds and evens.

1. With Cuisenaire rods, ask the question "Which rods can be matched with a two-piece, one-color train?" Record the results.

2. With sets of objects (pennies, perhaps), pair them up to see whether or not one is left. Count the objects and record the results.

3. Learn to identify odds and evens with any concrete objects.

B. Doubling games. Start with 1 and see how far you can go; start with 5, 3, 1/3, etc. Do this on graph paper, starting at upper right-hand corner.

C. Halving games. Start with 100, 80, etc. When do you have to stop? Do you really have to stop? Do you ever have to stop when you are doubling?

D. Make addition and multiplication tables for odds and evens. Test for commutativity, etc.

+	o	e
o		
e		

x	o	e
o		
e		

E. Try to learn to identify odds and evens in other bases. Use Dienes blocks to start. Are there patterns?

SEQUENCES

For lots of ideas of working with sequences on a number line, see Page, <u>Number Lines, Functions and Fundamental Topics</u>. Page's "cricket jumps" provide excellent drill in computational skills. They may be used only with positive whole numbers, or with fractions, decimals, all integers, etc. Some crickets always jump seven spaces forward, others jump three spaces backwards, etc. They start at varying places on the number line. Sometimes one child shows a few cricket jumps and another is asked to tell what kind of jumps are used. One can ask what size jumps can be used, starting from 0, to be sure of landing, for instance, on 36.

Similar work can be done without the number line on worksheets that ask for blanks to be filled to complete the pattern:

(1) 3, 7, 11, _____, _____, _____.

(2) 2, -3, -8, _____, _____, _____.

Much more complicated numbers and patterns can be used to fit the level of ability involved. It is fun for children to make worksheets for their friends to tackle.

The children can also experiment with other formats as well, such as graph paper (one-inch variety) and patterns created in squares. Give just a few clues and see if the children can fill in the rest.

			4						
									30
		53							
						87			

or

						66			
				45					
		24							
3									

or

	4								
				30					
						76			

or

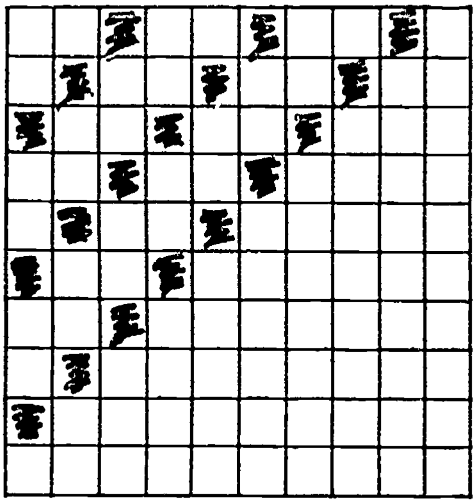
		5							
									39

or play with the multiplication tables by means of patterns. Lots of work in this area has been done by Rasmussen in her Math Lab for Children booklets, the Nuffield Mathematics Project's booklet Mathematics Begins, and by Biggs, Freedom to Learn.

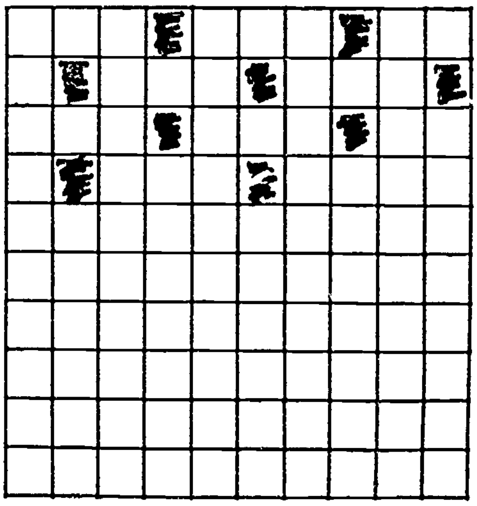
See more patterns on the next page.

Color in the pattern of the 3's (4's, 2's, etc.):

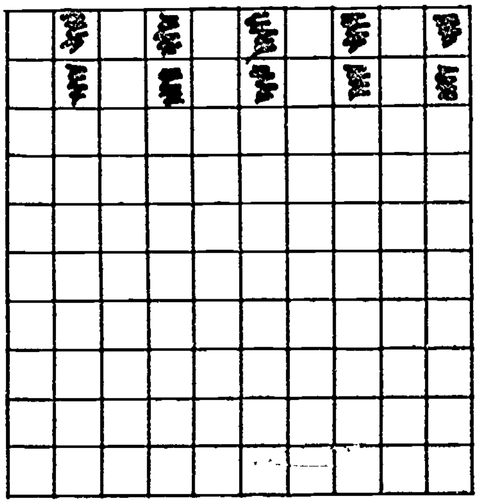
3's



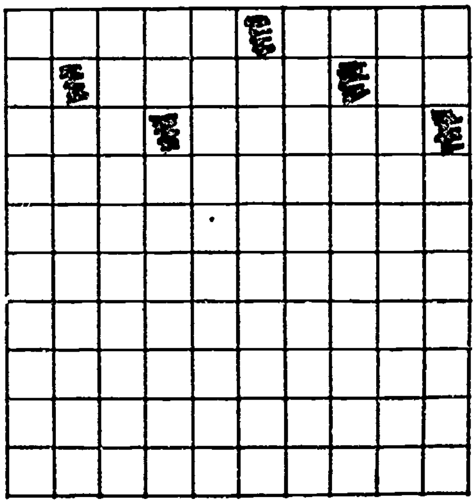
4's



2's



6's



There's no end to patterns. Try some of your own!

PATTERNS WITH NUMBERS

To have children identify prime (whole numbers with exactly two different factors) and composite numbers (more than two), it helps to use a table. A worksheet for the children to complete after much informal work with special trains begins to refine the idea of primes and composites. It might look something like this:

Special Trains

I can make exactly two one-color trains for these trains.	I can make more than two.

After the children have classified many rods, including lengths like (5 x O)* + 4 and so on, you may want them to give the rods their whole-number names. The completed sheet would look like this (see next page):

* "O" stands for an orange rod.

PRIMES		COMPOSITES	
Exactly two		More than two	
R	2	P	4
G	3	D	6
Y	5	N	8
K	7	E	9
(O + W)	11	O	10
(O + G)	13	O + R	12
(O + K)	17	O + P	14
(O + E)	19	O + Y	15
(2 x O) + W)	21	O + D	16
(2 x O) + G)	23	O + N	18
.	.	2 x O	20
.	.	(2 x O) + R	22
.	.	.	.
.	.	.	.

References

Davidson, Jessica, Using the Cuisenaire Rods

Fitzgerald et al., Laboratory Manual for Elementary Mathematics

University of Illinois Committee on School Mathematics, Stretchers and Shrinkers

OTHER TOPICS

Work with prime factorization, factors, multiples, least common multiples, greatest common factors, etc., can be done very well with rods, as described in the Cuisenaire book.

It is fun to make a 10 x 10 grid with the numbers 1 to 100 put in the squares and use colors to develop patterns of intervals, multiples, sums, etc.

The Sieve of Eratosthenes can, of course, be used to discover the primes.

See chapter 6 of Fitzgerald et al., Laboratory Manual for Elementary Mathematics, for a development of the Euclidian algorithm.

Make a graph of the multiplication tables. Use colors to show all kinds of things--multiples, factors, common factors, primes, composites, etc. (See Pearcy and Lewis, Experiments in Mathematics.)

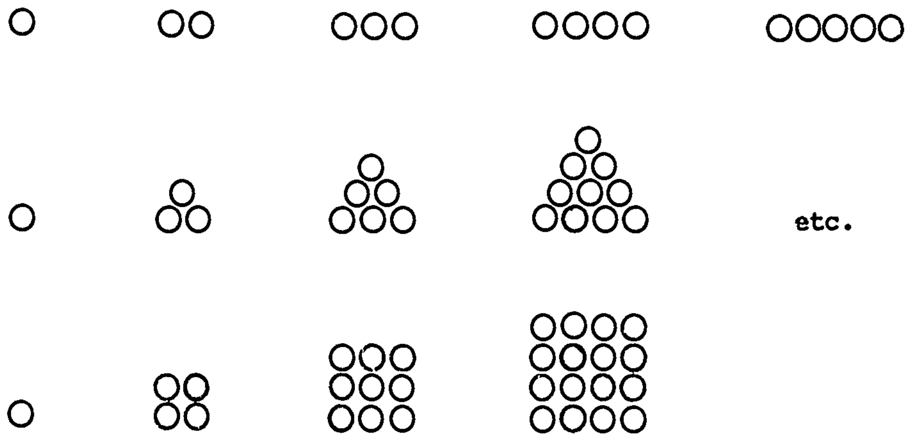
Work out tests for divisibility of whole numbers by 2, 3, 4, 5, etc.

What factors must be tried in order to be sure that 139, for instance, is prime? How can you be sure of always using the fewest "tests" in such a situation?

The Nuffield Mathematics Project Problems (both the Green and the Red sets) provide a great variety of interesting work with number patterns.

The Game of Lee is also an interesting challenge. The numbers 2 to 25 are written on the board. Player A picks one of the numbers (but is only allowed to pick one that has factors remaining on the board) and gets it for his score. That number is crossed out. Player B then picks (and crosses out all the factors of A's number), and gets the sum of them for his score. A and B continue to take turns, as above, until there is no longer available to A a number with factors on the board. The two then alternate, picking (and scoring) the remaining numbers. The higher score wins. It seems a good idea, when introducing the game, for the teacher to be A and for members of the class to be B. Usually after one such game most of the students will want to be A, will develop strategies, will try to figure the highest possible score, etc. The factors of the first 25 counting numbers will be very well known!

Using pennies, cubes, or marbles, make representations of numbers that are square, rectangular, triangular, etc. Make pyramids of the triangular ones, and cubes of the square ones. Count the marbles needed.



Classify numbers as to what shapes can be made:

Square	Rectangle	Triangle	Pyramid	Cube	Other

Shortest Routes (on the geoboard, allowing no diagonals).

Pascal's Triangle. Find as many patterns and peculiarities as possible about it. Find other sorts of things that also generate Pascal's triangle (such as arrangements of heads and tails, simple work in selecting committees, etc.).

BIBLIOGRAPHY

(Note: A [T] following an entry indicates a book intended primarily for teachers; a [C] indicates materials suitable for children; and [T, C] denotes materials suitable for both teachers and children. Certain books that are out of print are listed in the hope that they may be found in the library.)

Books

Bell, Stuart. Mathematics in the Making. 1. Pattern, Area and Perimeter; 2. Binary and Other Number Systems; 3. Looking at Solids; 4. Rotation and Angles; 5. Curves; 6. Scale Drawing and Surveying; 7. Transformation and Symmetry; 8. Networks; 9. All Sorts of Numbers; 10. Graphs; 11. Sets and Relations; 12. Statistics. Published in England, these booklets, which lend themselves especially to independent study and projects, are definitely for older children. [C]

Biggs, Edith, and James MacLean. Freedom to Learn. Don Mills, Ontario: Addison-Wesley Canada, Ltd., 1971. [T]

Brumfiel, Charles, Robert Eicholz, and Merrill Shanks. Algebra I. Palo Alto: Addison-Wesley, 1964. [T]

Charbonneau, Manon. Learning to Think in a Math Lab. Boston: National Association of Independent Schools, 1971. [T]

__________. Logic with Hidden Rods. Learning Center at Santa Fe, 1300 Canyon Rd., Santa Fe, N.M. 87501, 1973. [T]

__________. Work with the Sorting Box. Learning Center at Santa Fe, 1300 Canyon Rd., Santa Fe, N.M. 87501, 1973. [T]

Clarke, Mollie. Mollie Clarke Books: Symmetrical Shapes, A Dozen Eggs, Buttons, Shapes, What Is Inside?, Cakes and Candies, Sweets, Square Centimeters, A Box of Crayons, The Calendar, Houses, 20 Sticks, What Is Missing?, Dominoes, Numbers, A Group of Children, Beads, The Piggy Bank. Exeter, England: Wheaton, 1965-1973. A series of colorful paperback booklets that are excellent for very young children. [C]

Cochran, Beryl. An Approach to Place Value Using Beans and Beansticks. Education Development Center, 55 Chapel St., Newton, Mass. 02158, 1970. [T]

Cohen, Don. Inquiry in Math via Geoboards. New York: Walker, 1967. [T]

Davidson, Jessica. Using the Cuisenaire Rods: A Photo/Text Guide for Teachers. Cuisenaire Company of America, 12 Church St., New Rochelle, N.Y. 10805, 1969. [T]

Davidson, Patricia. Chip Trading Activities, Books 1-5 and Teacher's Guide. Scott Resources, Inc., Box 2121, Fort Collins, Colo. 80521, 1974. Excellent activities designed for use by children, but better used as a resource for ideas by the teacher. [T, C]

Davis, Robert B. Discovery in Mathematics: A Text for Teachers. Palo Alto: Addison-Wesley, 1963. [T]

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Dienes, Z. P. Building Up Mathematics. London: Hutchinson, 1960. [T]

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Fletcher, Harold. Mathematics for Schools: An Integrated Series--Level I (five- to seven-year-olds) and Level II (eight- to eleven-year-olds). London, England: Addison-Wesley, 1970. [C]

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Hull, William P. Attribute Games and Problems. New York: McGraw-Hill/Webster Division, 1968. [T]

Jacobs, Harold R. Mathematics: A Human Endeavor. San Francisco: W. H. Freeman, 1970. [T]

Johnson, Donovan, and William H. Glenn. Exploring Mathematics on Your Own. New York: Doubleday, 1961. [T]

Lawrence, E., T. R. Theakston, and N. Isaacs. Some Aspects of Piaget's Work. London, England: National Froebel Foundation, 1955. [T]

Marsh, T. G. Let's Explore Mathematics, Books 1-4, and Children Explore Mathematics (teacher's manual). New York: Arco Publishing Co., 1964-1968. Four books, graded in difficulty, not by grade levels; full of interesting mathematical topics to explore; better, perhaps, for seven- to nine-year olds. [C]

Minnemast Project. Booklets 1-30 (seven for kindergarten, seven for grade 1, eight for grade 2, four for grade 3, four for grade 4, plus a probability workbook). Minnemath Center, University of Minnesota, 720 Washington Ave., S.E., Minneapolis, Minn., 55414, n.d. The Minnemast Project was created for experimenting with relating mathematics to science in grades K-4. These booklets, the outcome of research, contain some very interesting topics and ideas but are very definitely oriented to the whole class's working together. [C]

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Nuffield Mathematics Project. (All of the following Nuffield publications, published since the mid-1960's, are distributed by John Wiley and Co., New York, N.Y.) Beginnings; Checking Up, I & II; Computation and Structure, 2-5; Computers and Young Children; Desk Calculators; Environmental Geometry; Graphs Leading to Algebra; How to Build a Pond; I Do, and I Understand; Into Secondary School; Logic; A Look Ahead; Mathematics Begins; Mathematics: The First Three Years; Mathematics: The Later Primary Years; Maths with Everything; Pictorial Representation; Primary Mathematics Report; Probability and Statistics; Problems: Green, Red, Purple; Shape and Size, 2-4; The Story So Far; Your Child and Mathematics. [T]

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Rasmussen, Lore. Math Lab for Children: Red Book, Orange Book, Green Book, Blue Book, Yellow Book, Purple Book. Learning Innovations Corp., 100-35 Metropolitan Ave., P.O. Box 430, Forest Hills, N.Y. 11375. An excellent group of mathematics workbooks for children ages six to eight, color-coded by difficulty or depth to which a topic is taken. [C]

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Tannenbaum, H., B. Tannenbaum, M. Stillman, and N. Stillman. Mapping. Beginner Science Series, Grade 5, Unit 5. New York: McGraw-Hill/Webster Division, 1966-67. From a series of workbooks on different mathematical and scientific topics, this one has proven most interesting and useful with eight- and nine-year-olds. [C]

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Whittaker, Dora. Move In On Math, Books 1-4, and Teacher's Notes. Cuisenaire Company of America, 12 Church St., New Rochelle, N.Y. 10805, n.d. This sequential set of books, published in England, is for the elementary grades (not grades 1-4, but to be expanded across the years for six-through eleven-year-olds). [C]

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Yeomans, Edward. Education for Initiative and Responsibility. Boston: National Association of Independent Schools, 1968. [T]

Games

The following list is by no means definitive, but it gives good basic games with which to start building a more complete stock.

Quinto
Stocks and Bonds
Numble
Heads Up
Cover Up
Kalah
Dominoes
Triminoes
Pentaminoes
Chip Trading
Vector
Hex
Tower of Hanoi
Insanity Blocks
Nine-Piece Puzzle
Monopoly
Tuf
Competitive Fractions
Hi-Q
Krypto
Fraction Bars
Tangrams
Racko
Chess
Snow Crystals
Altair Designs